Introduction to the theory of hyperbolic equations: hyperbolic systems

Luca Bonaventura

POLITECNICO
MILANO 1863
DIPARTIMENTO DI MATEMATICA

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Outline of the lecture

Linear hyperbolic systems

Nonlinear systems of conservation laws

The Euler equations of gas dynamics

Riemann problem for Euler equations

Key concepts introduced in the second lecture

- **Linear hyperbolic systems and their mathematical formulation**
- **Nonlinear systems of conservation laws**
- **The Euler equations as a hyperbolic system**
- **The Riemann problem for the Euler equations and its fundamental solutions**

Linear hyperbolic systems

Linear first order systems

- **Unknowns:** d **functions of space and time** $c_i(x,t), i=1,\dots,d$
- **Coupling coefficients:** $d \times d$ **constants** $a_{ij}, i,j=1,\dots,d$
- **Express in vector notation: coupled set of advection equations**

$$\mathbf{c}(x,t) = \begin{bmatrix} c_1(x,t) \\ \cdots \\ \cdots \\ c_d(x,t) \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ a_{21} & \cdots & a_{2d} \\ \cdots & \cdots & \cdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix}$$

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{c}}{\partial x} = \mathbf{0}$$

Linear hyperbolic systems (1)

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{c}}{\partial x} = 0 \qquad (1)$$

Hyperbolicity

System (1) is hyperbolic if A is diagonalized by some invertible matrix T and it has real eigenvalues λ_i; it is strictly hyperbolic if all eigenvalues are real and distinct: $\lambda_i \neq \lambda_j \quad \text{for} \quad i \neq j$

$$\mathbf{A} = \mathbf{T\Lambda T}^{-1} \qquad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \lambda_d \end{bmatrix} \qquad \lambda_i \in \mathbf{R}$$

Linear algebra reminder

The fact that A is diagonalized by some invertible matrix T is equivalent to:

- **for each** $\lambda_i \quad i = 1, \ldots, d$ **there is a vector** $\mathbf{t}_i$ **such that**

$$\mathbf{A}\mathbf{t}_i = \lambda_i \mathbf{t}_i \qquad i = 1, \ldots, d$$

- **AND** $\quad \mathbf{t}_i \quad i = 1, \ldots, d$ **are linearly independent and**

$$\mathbf{T} = [\mathbf{t}_1, \ldots, \mathbf{t}_d].$$

Linear hyperbolic systems (2)

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{T}\boldsymbol{\Lambda}\mathbf{T}^{-1}\frac{\partial \mathbf{c}}{\partial x} = \mathbf{0} \quad \rightarrow \quad \mathbf{T}^{-1}\frac{\partial \mathbf{c}}{\partial t} + \boldsymbol{\Lambda}\mathbf{T}^{-1}\frac{\partial \mathbf{c}}{\partial x} = \mathbf{0}$$

$$\mathbf{e} = \mathbf{T}^{-1}\mathbf{c} \quad \rightarrow \quad \frac{\partial \mathbf{e}}{\partial t} + \boldsymbol{\Lambda}\frac{\partial \mathbf{e}}{\partial x} = \mathbf{0}$$

$$\frac{\partial e_i}{\partial t} + \lambda_i \frac{\partial e_i}{\partial x} = 0 \quad i = 1, \dots, d$$

- **After change of variables: uncoupled set of advection equations in terms of the characteristic variables e**
- **Solution procedure: diagonalize and then exploit exact solution of the scalar advection equation**

Linear hyperbolic systems (3)

To determine exact solution:

- **transform initial data $\mathbf{c}_0$ into $\mathbf{e}_0 = \mathbf{T}^{-1}\mathbf{c}_0$**
- **exact solutions in terms of the transformed, uncoupled variables**

$$e_i(x,t) = e_{i,0}(x - \lambda_i t), \quad i = 1, \dots, d$$

- **transform back to original variables**

$$\mathbf{c}(x,t) = \mathbf{T}\mathbf{e}(x,t) = \sum_{i=1}^{d} e_{i,0}(x - \lambda_i t)\mathbf{t}_i$$

- **solution is superposition of d independent simple waves**

Linear hyperbolic systems (4)

Initial and boundary value problem

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{c}}{\partial x} = \mathbf{0} \qquad x \in [0, L], \quad t \in [0, T]$$

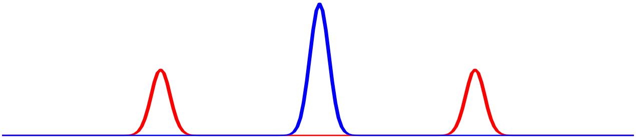
- **Solution on a bounded interval $x \in [0, L]$: some information is needed about $\mathbf{c}(x, t)$ at the boundary upstream with respect to $\lambda_i \quad i = 1, \ldots, d$ for $t \in [0, T]$**
- **Suppose $\lambda_i \geq 0 \qquad i = 1, \ldots, k \qquad \lambda_i \leq 0 \qquad i = k+1, \ldots, d$**
- **Mathematical formulation as initial and boundary value problem: assuming that $\mathbf{c}(x, 0) = \mathbf{c}_0(x)$ is known for $t = 0$, $e_i(0, t) = g_i(t) \;\; i = 1, \ldots, k \;\; e_i(L, t) = g_i(t) \;\; i = k+1, \ldots, d$ are known for $t \in [0, T]$, determine $\mathbf{c}(x, t)$ at later times**

Examples: linear channel flow (1)

$$
\begin{aligned}
\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} &= 0 \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} &= 0
\end{aligned}
$$

- g **gravity acceleration**, H **constant mean fluid depth**, U **constant mean velocity**
- h, u **small perturbations of uniform motion in a channel with mean fluid depth** H **and mean velocity** U

Let's have a look



- **Initial condition: still water $U=0, u=0$, small perturbation in free surface of the liquid**
- **System of 2 equations: 2 waves propagating with speeds $\pm\sqrt{gH}$**
- **No change in shape: regularity of the initial data is preserved**

Nonlinear systems of conservation laws

Nonlinear systems of conservation laws

$$\frac{\partial \mathbf{c}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{c}) = \mathbf{0}$$

$$\mathbf{c}(x,t) = \begin{bmatrix} c_1(x,t) \\ \dots \\ \dots \\ c_d(x,t) \end{bmatrix} \qquad \mathbf{f}(\mathbf{c}) = \begin{bmatrix} f_1(c_1,\dots,c_d) \\ \dots \\ \dots \\ f_d(c_1,\dots,c_d) \end{bmatrix}$$

Coupled set of nonlinear conservation laws, mathematical formulation of the conservation principles for d quantities $c_1,\dots,c_d$

Nonlinear hyperbolic systems (1)

$$\frac{\partial \mathbf{c}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{c}) = \frac{\partial \mathbf{c}}{\partial t} + \mathbf{f}'(\mathbf{c})\frac{\partial \mathbf{c}}{\partial x} = \mathbf{0}$$

$$\mathbf{f}'(\mathbf{c}) = \begin{bmatrix} \frac{\partial f_1}{\partial c_1} & \frac{\partial f_1}{\partial c_2} & \dots & \frac{\partial f_1}{\partial c_d} \\ \frac{\partial f_2}{\partial c_1} & \frac{\partial f_2}{\partial c_2} & \dots & \frac{\partial f_2}{\partial c_d} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_d}{\partial c_1} & \dots & \dots & \frac{\partial f_d}{\partial c_d} \end{bmatrix}$$

- **Physical interpretation: coupled set of nonlinear advection equations, advective form of the system of conservation laws**
- **Mathematical definition: system is hyperbolic if $\mathbf{f}'(\mathbf{c})$ is diagonalizable with real eigenvalues, strictly hyperbolic if it has real and distinct eigenvalues**

Nonlinear hyperbolic systems (2)

Initial and boundary value problem

$$\frac{\partial \mathbf{c}}{\partial t} + \mathbf{f}'(\mathbf{c})\frac{\partial \mathbf{c}}{\partial x} = \mathbf{0} \qquad x \in [0, L], \quad t \in [0, T]$$

- **Solution on a bounded interval $x \in [0, L]$: some information is needed about $\mathbf{c}(x, t)$ at the boundary upstream with respect to $\lambda_i(\mathbf{c}(0, t)), \lambda_i(\mathbf{c}(L, t)) \quad i = 1, \ldots, d$ eigenvalues of $\mathbf{f}'(\mathbf{c})$ for $t \in [0, T]$**
- **Suppose**
  $\lambda_i(\mathbf{c}(0, t)) \geq 0 \quad i = 1, \ldots, k \quad \lambda_i(\mathbf{c}(0, t)) \leq 0 \quad i = k+1, \ldots, d$
  $\lambda_i(\mathbf{c}(L, t)) \leq 0 \quad i = 1, \ldots, k \quad \lambda_i(\mathbf{c}(L, t)) \geq 0 \quad i = k+1, \ldots, d$
- **Mathematical formulation as initial and boundary value problem: assuming that $\mathbf{c}(x, 0) = \mathbf{c}_0(x)$ is known for $t = 0$ and $e_i(0, t) = g_i(t) \quad i = 1, \ldots, k \;\; e_i(L, t) = g_i(t) \quad i = k+1, \ldots, d$ are known for $t \in [0, T]$, determine $\mathbf{c}(x, t)$ at later times**

Nonlinear hyperbolic systems (3)

- **Nonlinear dependence of eigenvalues and eigenvectors of $\mathbf{f}'(\mathbf{c})$ on the state $\mathbf{c}$:**

$$\mathbf{f}'(\mathbf{c})\mathbf{t}_i(\mathbf{c}) = \lambda_i(\mathbf{c})\mathbf{t}_i(\mathbf{c}) \quad i = 1, \dots, d$$

- $\lambda_i(\mathbf{c})$ **is called characteristic field**
- **Integral curves in state space: $\bar{\mathbf{c}}(s)$ such that**

$$\bar{\mathbf{c}}'(s) = \alpha(s)\mathbf{t}_i(\bar{\mathbf{c}}(s)) \ \ \text{for some} \quad i = 1, \dots, d$$

- **Riemann invariant associated to integral curve: function $w(\mathbf{c})$ such that for some $i = 1, \dots, d$**

$$\nabla w \cdot \mathbf{t}_i = 0$$

Nonlinear hyperbolic systems (4)

- **Riemann invariants are constant along the corresponding integral curves:**

$$\frac{d}{ds}w(\bar{\mathbf{c}}(s)) = \nabla w \cdot \bar{\mathbf{c}}'(s) = \alpha(s)\nabla w \cdot \mathbf{t}_i(\bar{\mathbf{c}}(s)) = 0$$

- **Variation of characteristic fields along integral curves:**

$$\frac{d}{ds}\lambda_i(\bar{\mathbf{c}}(s)) = \nabla \lambda_i \cdot \bar{\mathbf{c}}'(s) = \alpha(s)\nabla \lambda_i \cdot \mathbf{t}_i(\bar{\mathbf{c}}(s))$$

- **Genuinely nonlinear characteristic fields** $\nabla\lambda_i \cdot \mathbf{t}_i(\bar{\mathbf{c}}(s)) \neq 0:$ **charateristic fields vary monotonically along integral curves**
- **Linearly degenerate characteristic fields** $\nabla\lambda_i \cdot \mathbf{t}_i(\bar{\mathbf{c}}(s)) = 0:$ **the characteristic field is a Riemann invariant**

Nonlinear hyperbolic systems (5)

- General solution is extremely complex and can only be approximated numerically
- Special solutions are simple waves: $\mathbf{c}(x,t) = \bar{\mathbf{c}}(s(x,t))$, where $\bar{\mathbf{c}}(s)$ is an integral curve associated to $\mathbf{t}_i$ and $s(x,t)$ some regular function of space and time
- Impose conditions for simple wave $\mathbf{c}(x,t)$ to be solution yields scalar conservation law

$$\frac{\partial s}{\partial t} + \lambda_i(\bar{\mathbf{c}}(s(x,t)))\frac{\partial s}{\partial x} = 0$$

- Riemann problem with piecewise constant $\mathbf{c}_l, \mathbf{c}_r$ initial value can be formulated also for nonlinear hyperbolic systems

Simple waves solutions of Riemann problems (1)

- **Shocks**: necessary **conditions** to have a shock solution with speed s are the **Rankine-Hugoniot** conditions

$$\mathbf{f}(\mathbf{c}_r) - \mathbf{f}(\mathbf{c}_l) = s(\mathbf{c}_r - \mathbf{c}_l)$$

- RH conditions are not **sufficient** to determine the shock solution completely: also knowledge of some **Riemann invariants** is required
- **Rarefaction wave** solutions can be built in a **similar** way as scalar case for **genuinely nonlinear** characteristic fields, assuming $s(x,t) = (x - x_0)/t$ in simple wave definition, resulting condition is

$$\bar{\mathbf{c}}'(s) = \frac{\mathbf{t}_i(\bar{\mathbf{c}}(s))}{\nabla \lambda_i(\bar{\mathbf{c}}(s)) \cdot \mathbf{t}_i(\bar{\mathbf{c}}(s))}$$

Simple waves solutions of Riemann problems (2)

- **Contact waves: simple wave solution associated to linearly degenerate characteristic fields**

$$\nabla \lambda_i(\bar{\mathbf{c}}(s)) \cdot \mathbf{t}_i(\bar{\mathbf{c}}(s)) = 0$$

- **Linear degeneracy implies $\lambda_i(\bar{\mathbf{c}}(s(x,t)))$ is constant along the characteristic line**
- **Riemann problem solution is simple (analytically): discontinuity propagates at speed $\lambda_i(\mathbf{c}_l) = \lambda_i(\mathbf{c}_r)$**
- **General solution of Riemann problem for $d \times d$ system will consist of combination of d simple waves associated to each different characteristic field**

Systems of conservation laws with source terms

$$\frac{\partial \mathbf{c}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{c}) = \mathbf{s}(\mathbf{c}, x, t)$$

Source may represent:

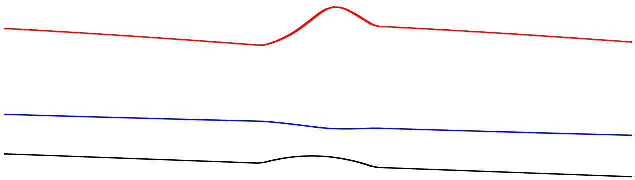
- **changes in geometry: variable sized ducts, compliant walls (arteries)**
- **external sources or sinks of mass or momentum: friction**
- **chemical reactions and changes in state: combustion, detonation**

Examples: nonlinear channel flow

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = s(x,t)$$
$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{h} + \frac{gh^2}{2}\right) = -gh\frac{\partial b}{\partial x}$$

- h **fluid depth,** $q = uh$ **momentum (discharge),** b **channel bed depth,** s **water source**
- **Homogeneous shallow fluid in one dimensional channel with rectangular cross section**
- **Shallow water equations or de Saint Venant equations: basic model of river hydraulics, analogous to the equations of isentropic gas**

Let's have a look



- **Nonlinear shallow water equations, steady state solution with constant momentum, solved semi-analytically**
- **Source term in momentum equation: channel slope and obstacle, friction**
- **Water elevation (blue), velocity (red), bed depth (black).**

The Euler equations of gas dynamics

The Euler equations

- Mathematical formulation of fundamental conservation principles in classical physics
- Model for short time scale evolution of an ideal gas
- Basic tool in aerodynamics, aeronautic engineering, combustion theory, numerical weather prediction

Fundamental definitions

- ρ **mass density**, u **velocity**, $q = \rho u$ **momentum**
- p **pressure**, e **specific internal energy**, $E = \rho(u^2/2 + e)$ **total energy**
- c_p, c_v **specific heats**, $\gamma = \frac{c_p}{c_v}$ **adiabatic exponent (ratio of specific heats)**
- **Equation of state (ideal gas)** $p = e(\gamma - 1)\rho = RT\rho$
- $s = c_v \log(p/\rho^\gamma)$ **entropy**, $H = (E + p)/\rho$ **total enthalpy**, $h = e + p/\rho$ **specific enthalpy**
- **Sound speed**

$$a = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_{entropy}} = \sqrt{\frac{\gamma p}{\rho}}$$

Fundamental conservation laws

Consider a gas contained in a straight tube of unit area cross section: for each portion (a, b) of the tube, impose

- **Conservation of mass**

$$\int_a^b \rho(x,t) \quad dx$$

- **Conservation of (linear) momentum**

$$\int_a^b q(x,t) \quad dx = \int_a^b \rho(x,t)u(x,t) \quad dx$$

- **Conservation of energy**

$$\int_a^b E(x,t) \quad dx$$

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The Euler equations (1)

Formulation using primitive variables ρ, u, p :

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0$$
$$\frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x}\left(\rho u^2 + p\right) = 0$$
$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[(E + p)u\right] = 0$$
$$p = (\gamma - 1)\left(E - \rho\frac{u^2}{2}\right)$$

The Euler equations (2)

Formulation using conserved quantities ρ, q, E:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{3-\gamma}{2}\frac{q^2}{\rho} + (\gamma-1)E\right) = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left[\gamma\frac{q}{\rho}E - \frac{\gamma-1}{2}\frac{q^3}{\rho^2}\right] = 0$$

$$p = (\gamma-1)\left(E - \frac{q^2}{2\rho}\right)$$

The Euler equations (3)

In vector notation, $\mathbf{c} = [\rho, q, E]^T$,

$$\mathbf{f}(\mathbf{c}) = \left[q,\ \frac{3-\gamma}{2}\frac{q^2}{\rho} + (\gamma-1)E,\ \gamma\frac{q}{\rho}E - \frac{\gamma-1}{2}\frac{q^3}{\rho^2} \right]^T$$

$$\mathbf{f}'(\mathbf{c}) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\left(\frac{q}{\rho}\right)^2 & (3-\gamma)\frac{q}{\rho} & (\gamma-1) \\ -\gamma\frac{qE}{\rho} + (\gamma-1)\left(\frac{q}{\rho}\right)^3 & \gamma\frac{E}{\rho} - \frac{3(\gamma-1)}{2} & \gamma\frac{q}{\rho} \end{bmatrix}$$

The Euler equations (4)

- System is always strictly hyperbolic: eigenvalues of $\mathbf{f}'(\mathbf{c})$

$$\lambda_1 = \frac{q}{\rho} - \sqrt{\frac{\gamma p}{\rho}} = u - a \quad \lambda_2 = \frac{q}{\rho} = u \quad \lambda_3 = \frac{q}{\rho} + \sqrt{\frac{\gamma p}{\rho}} = u + a$$

- Eigenvectors of $\mathbf{f}'(\mathbf{c})$:

$$\mathbf{t}_1 = \begin{bmatrix} 1 \\ u - a \\ H - ua \end{bmatrix} \quad \mathbf{t}_2 = \begin{bmatrix} 1 \\ u \\ \frac{u^2}{2} \end{bmatrix} \quad \mathbf{t}_3 = \begin{bmatrix} 1 \\ u + a \\ H + ua \end{bmatrix}$$

- Characteristic fields: one is linearly degenerate, two are genuinely nonlinear

$$\nabla \lambda_1 \cdot \mathbf{t}_1 \neq 0, \quad \nabla \lambda_2 \cdot \mathbf{t}_2 = 0, \quad \nabla \lambda_3 \cdot \mathbf{t}_3 \neq 0$$

The Euler equations (5)

Homogeneity property of the Euler equations: $\mathbf{f}(\mathbf{c}) = \mathbf{f}'(\mathbf{c})\mathbf{c}$

$$\mathbf{f}(\mathbf{c}) = \mathbf{f}'(\mathbf{c})\mathbf{c} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\left(\frac{q}{\rho}\right)^2 & (3-\gamma)\frac{q}{\rho} & (\gamma-1) \\ -\gamma\frac{qE}{\rho} + (\gamma-1)\left(\frac{q}{\rho}\right)^3 & \gamma\frac{E}{\rho} - \frac{3(\gamma-1)}{2} & \gamma\frac{q}{\rho} \end{bmatrix} \begin{bmatrix} \rho \\ q \\ E \end{bmatrix}$$

Important for Flux Vector Splitting numerical methods

The Euler equations: useful exercises

- **Rewrite in terms of primitive variables** ρ, u, p, T
- **Rewrite in terms of non dimensional primitive variables** $\rho/\rho_{ref}\,, u/u_{ref}\,, p/p_{ref}\,, T/T_{ref}$ **and Mach number** $Ma = u/a$
- **Rewrite for tube with variable cross section in space**
- **Rewrite for tube with variable cross section in space and time: compliant walls**

The Riemann problem for the Euler equations

A special initial and boundary value problem

Riemann problem

Consider the Euler equations for $x \in [0, L]$ with initial condition

$$\mathbf{c}_0(x) = \mathbf{c}_l \quad x \leq x_0 \qquad \mathbf{c}_0(x) = \mathbf{c}_r \quad x > x_0 \qquad x \in (0, L)$$

and boundary conditions $\mathbf{c}(0, t) = \mathbf{c}_l$, $\mathbf{c}(L, t) = \mathbf{c}_r$.

- Exact solution can be computed in some cases: useful to understand physics and to test numerical methods
- Also useful to build numerical methods: Godunov methods

Structure of the Riemann problem solution

- Solution must be superposition of shock waves, rarefaction waves, contact waves
- There is always a contact wave travelling at intermediate velocity between two waves of other type, since

$$\nabla \lambda_2 \cdot \mathbf{t}_2 = 0 \qquad \lambda_1 < \lambda_2 < \lambda_3$$

- Four basic possibilities: SCS, SCR, RCS, RCR
- Use Rankine Hugoniot conditions to understand if initial data are compatible with shock solutions for waves 1,3. If not, compute rarefaction wave for wave that cannot be shock, then compute shock wave
- For computation, use the fact that Riemann invariants are constant for simple wave solutions

Riemann invariants for Euler equations

Functions of $w(\mathbf{c}) = w(\rho, q, E)$ that are constant along some simple wave because $\nabla w \cdot \mathbf{t}_i = 0$

- **Wave 1:**

$$s = c_v \log\left(p/\rho^\gamma\right) \qquad u + \frac{2a}{\gamma - 1} = \frac{q}{\rho} + \sqrt{\frac{\gamma p}{\rho}}$$

- **Wave 2:**

$$u = \frac{q}{\rho} \qquad p = (\gamma - 1)\left(E - \frac{q^2}{2\rho}\right)$$

- **Wave 3:**

$$s = c_v \log\left(p/\rho^\gamma\right) \qquad u - \frac{2a}{\gamma - 1} = \frac{q}{\rho} - \sqrt{\frac{\gamma p}{\rho}}$$

- **Difficult exercise: check that these are Riemann invariants for the respective waves**

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Shock tube: a special Riemann problem

Shock tube problem

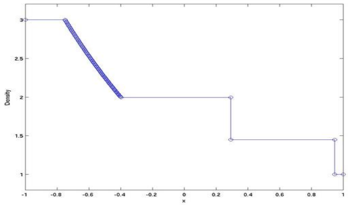
Consider the Euler equations for $x \in [0, L]$ with initial condition

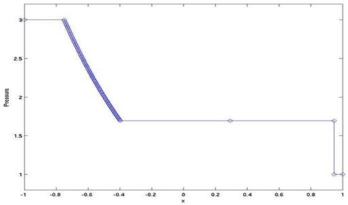
$$\mathbf{c}_l = [\rho_l, 0, E_l]^T \quad x \leq x_0 \qquad \mathbf{c}_r = [\rho_r, 0, E_r]^T \quad x > x_0$$

and boundary conditions $\mathbf{c}(0, t) = \mathbf{c}_l$, $\mathbf{c}(L, t) = \mathbf{c}_r$.

- Zero initial velocity, two tube sections separated by a wall
- With zero velocity, $E = p/(\gamma - 1)$: initial datum is usually defined in terms of pressure
- For shallow water equations, also known as dam break problem

Let's have a look





- **Exact solution of Sod shock tube problem, initial data**

$$\mathbf{c}_l = [3, 0, 3/(\gamma - 1)]^T \quad x \leq 0 \qquad \mathbf{c}_r = [1, 0, 1/(\gamma - 1)]^T \quad x > 0$$

- **Rarefaction wave associated to characteristic field 1, shock wave associated to characteristic field 3**
- **Contact wave associated to characteristic field 2: pressure is constant across contact**

Summary of the second lecture

- Linear hyperbolic systems and their mathematical formulation
- Nonlinear systems of conservation laws
- The Euler equations as a strictly hyperbolic system
- The Riemann problem for the Euler equations and its fundamental solutions