Introduction to the theory of hyperbolic equations: scalar problems

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Outline of the lecture

Linear advection

Classical and weak solutions

Nonlinear conservation laws

Riemann problem

Key concepts introduced in the first lecture

- **The advection equation and its mathematical formulation**
- **Classical (differentiable) and weak (non differentiable) solutions**
- **Nonlinear conservation laws**
- **The Riemann problem and its fundamental solutions**

Linear advection

The linear advection equation (1)

$$\frac{\partial c}{\partial t} + a\frac{\partial c}{\partial x} = 0 \qquad x \in \mathbf{R}$$

- **Physical interpretation:** a **advection velocity,** $c(x,t)$ **concentration of a chemical species transported by an ideal fluid without viscosity**
- **Mathematical formulation as initial value problem: suppose** $c(x,0) = c_0(x)$ **is known at** $t = 0$**, determine** $c(x,t)$ **at later times**

The linear advection equation (2)

If $c_0(x) \in \mathcal{C}^1(\mathbf{R})$ (the initial value is differentiable), there is a unique solution given by

$$c(x,t) = c_0(x - at)$$

Proof

$$\frac{\partial c}{\partial t} = -a c_0'(x - at) \qquad \frac{\partial c}{\partial x} = c_0'(x - at)$$

- **Physical interpretation of the solution: initial profile is transported at speed a by the fluid**
- **For $a > 0$: solution at x is determined by initial value upstream of x**

The linear advection equation (3)

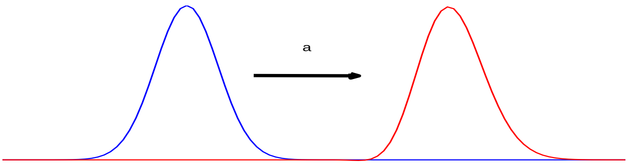
Initial and boundary value problem

$$\frac{\partial c}{\partial t} + a\frac{\partial c}{\partial x} = 0 \qquad a > 0 \qquad x \in [0, L], \quad t \in [0, T]$$

- **Solution on a bounded interval** $x \in [0, L]$ **: some information is needed about** $c(0, t)$ **for** $t \in [0, T]$
- **Mathematical formulation as initial and boundary value problem: assuming that** $c(x, 0) = c_0(x)$ **is known for** $t = 0$ **and** $c(0, t) = g(t)$ **is known for** $t \in [0, T]$**, determine** $c(x, t)$ **at later times**
- **Solution at** (x, t) **is:**

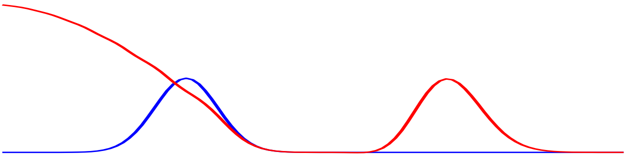
$$c(x, t) = c_0(x - at) \quad t < x/a \qquad c(x, t) = g(t - x/a) \quad t > x/a$$

Let's have a look: no inflow at boundary



- **No change in shape, no change in maximum or minimum value: maximum principle**
- **No change in regularity: smoothness of the initial data is preserved**

Let's have a look: inflow at boundary



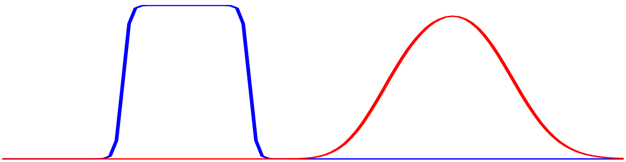
- Solution can change in shape depending on boundary condition
- After $T = L/a$, the initial data have no impact on the solution

The advection diffusion equation

$$\frac{\partial c}{\partial t} + a\frac{\partial c}{\partial x} = b\frac{\partial^2 c}{\partial x^2} \qquad x \in [0, L], \quad t \in [0, T]$$

- **Physical interpretation:** a **advection velocity,** $c(x,t)$ **concentration of a chemical species transported by a fluid with viscosity** b
- **Mathematical formulation as initial and boundary value problem: boundary conditions at both ends of the interval must be provided**
- **Solution can be computed analytically only in special cases (for example, impose periodic boundary conditions and use Fourier series expansions)**

Let's have a look



- **Physical interpretation: initial profile is transported at speed a by the fluid but also expands through the fluid because of microscopic diffusion processes**
- **Initial profile is smoothed: spatial derivatives become smaller**

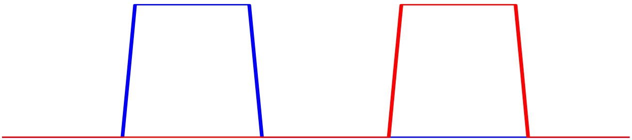
Some classification

- $a \neq 0, b = 0$: **pure advection, prototype of hyperbolic equations**
- $a = 0, b \neq 0$: **pure diffusion, prototype of parabolic equation**
- $a, b \neq 0$: **advection diffusion equation, neither strictly hyperbolic nor parabolic, both kinds of phenomena are present, phenomenology is ruled by**

Peclet number $$Pe = \frac{aL}{b}$$

Classical and weak solutions

Generalization of the solution concept



- **Even when the initial value is not differentiable $c_0(x) \notin \mathcal{C}^1(\mathbf{R})$ $c(x,t) = c_0(x - at)$ is still defined (even if $c_0(x) \notin \mathcal{C}^0(\mathbf{R})$!!)**
- **Physical interpretation: a chemical species with very irregular distribution is transported by the flow**
- **Mathematical problem: a new concept of solution is needed**

Weak solutions

The function $c(x, s)$ is a weak solution of the advection equation for $s \in [0, t]$ with initial data $c_0(x)$ and boundary data $c(0, s) = g_0(s)$, $c(L, s) = g_L(s)$ if for any $\phi \in \mathcal{C}^\infty([0, L] \times [0, t])$ it holds

$$\int_0^L c(x,t)\phi(x,t)\ dx - \int_0^L c_0(x)\phi(x,0)\ dx + \int_0^t a g_L(s)\phi(L,s)\ ds - \int_0^t a g_0(s)\phi(0,s)\ ds - \int_0^t \int_0^L \left[c\frac{\partial \phi}{\partial s} + ac\frac{\partial \phi}{\partial x} \right]\ dx\ ds = 0$$

- $\phi \in \mathcal{C}^\infty([0, L] \times [0, T])$: **test functions, depend both on space and time, differentiable infinitely many times**
- **Complicated mathematical definition that extends the conventional one**

Classical solutions are also weak solutions

Proof:

$$\frac{\partial c}{\partial s} + a\frac{\partial c}{\partial x} = 0 \qquad \textbf{implies} \qquad \int_0^t \int_0^L \phi \left[\frac{\partial c}{\partial s} + a\frac{\partial c}{\partial x}\right] \ dxds = 0$$

for any $\phi \in \mathcal{C}_0^1([0,L])$. **Then it follows (integration by parts)**

$$\phi\left[\frac{\partial c}{\partial s} + a\frac{\partial c}{\partial x}\right] = \left[\frac{\partial \phi c}{\partial s} + a\frac{\partial \phi c}{\partial x}\right] - c\left[\frac{\partial \phi}{\partial s} + a\frac{\partial \phi}{\partial x}\right].$$

Imposing initial and boundary conditions it follows

$$\int_0^L c(x,t)\phi(x,t) \ dx - \int_0^L c_0(x)\phi(x,0) \ dx$$
$$+ \int_0^t g_L(s)\phi(L,s) \ ds - \int_0^t g_0(s)\phi(0,s) \ ds$$
$$- \int_0^t \int_0^L \left[c\frac{\partial \phi}{\partial s} + ac\frac{\partial \phi}{\partial x}\right] \ dx \ ds = 0$$

An example of weak solution that is not a classical solution

- $c_0(x) = \chi_{(b,c)}(x)$: **initial data characteristic function of interval** (b, c), **intuitively** $c_0(x - at)$ **should be a solution**
- **take** $g_L = g_0 = 0$, $\phi = 1$ **in definition of weak solution and check that** $c(x, t) = c_0(x - at)$ **satisfy it!**

Weak solutions as viscosity solutions

Advection equation with viscosity

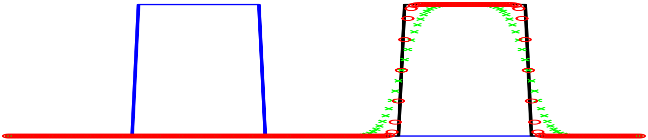
$$\frac{\partial c_\epsilon}{\partial t} + a\frac{\partial c_\epsilon}{\partial x} = \epsilon\frac{\partial^2 c_\epsilon}{\partial x^2}$$

- **Due to the parabolic term, solutions of advection diffusion equations are always differentiable (classical solutions)**
- **Weak solutions can be obtained as vanishing viscosity limit**

$$c(x,t) = \lim_{\epsilon\to 0} c_\epsilon(x,t).$$

- **Unpractical to do it numerically all the time: direct numerical approximation of weak solutions is essential**

Let's have a look



- Very **accurate** numerical solution obtained by separation of variables and **Fourier series** expansion
- **Decreasing** values of viscosity: convergence to weak **inviscid** solution

Nonlinear conservation laws

Nonlinear conservation laws

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} f(c) = 0$$

- **A huge number of physical phenomena are modeled by conservation laws, that express mathematically conservation principles for extensive quantities: c conserved quantity, f flux of the conserved quantity**
- **Mathematical formulation: define initial and boundary value problem assigning initial and boundary conditions**
- **Generalization of linear advection: $f(c) = ac$.**

Burgers equation

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(\frac{c^2}{2}\right) = 0$$

- **Simplest non trivial, non linear conservation law: derived from conservation of momentum for fluid flow in extremely simplified conditions**
- **Analytical solution is known in several cases: useful as benchmark for numerical methods**
- **Viscous Burgers equation:**

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x}\left(\frac{c^2}{2}\right) = \epsilon \frac{\partial^2 c}{\partial x^2}$$

Why conservation laws?

$$
\begin{aligned}
\frac{d}{dt}\int_a^b c \; dx &= \int_a^b \frac{\partial c}{\partial t} \; dx \\
&= -\int_a^b \frac{\partial}{\partial x} f(c) \; dx = -\left[f(c(b,t)) - f(c(a,t))\right] \\
&= f(c(a,t)) - f(c(b,t))
\end{aligned}
$$

- **The total amount of quantity c in the interval $[a, b]$ only changes due to the fluxes through the boundaries**
- **If the flux is zero at the boundaries, the total amount of quantity c is conserved**

MOX

More general case

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} f(c) = s(c, x, t)$$

- **Conservation law with source term** $s = s(c, x, t)$
- **More general balance law:**

$$\frac{d}{dt} \int_a^b c \ dx = f(c(a, t)) - f(c(b, t)) + \int_a^b s(c(x, t), x, t) \ dx$$

The total amount of quantity c **in the interval** $[a, b]$ **changes due to the fluxes through the boundaries and due to processes at the interior of the interval**

Nonlinear conservation laws as nonlinear advection

Advective form of conservation law

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} f(c) = \frac{\partial c}{\partial t} + f'(c)\frac{\partial c}{\partial x} = 0$$

- **Physical interpretation: c quantity transported with velocity $f'(c)$ that depends on the value of c itself**
- **Mathematical interpretation: as long as the spatial derivatives of the solution are well defined, the two formulations are equivalent**

Characteristic curves

Assume that solution $c(x,t)$ exists and is differentiable.

A characteristic curve is the solution $x(t)$ of the ODE

$$\frac{dx}{dt} = f'(c(x(t),t)) \qquad x(0) = x_0 \in [0,L]$$

- **For each point x_0, standard ODE theory guarantees that only one $x(t)$ exists (at least for short intervals of time)**
- **Solutions are constant along characteristics:**

$$\frac{d}{dt}c(x(t),t) = \frac{\partial c}{\partial t} + \frac{dx}{dt}\frac{\partial c}{\partial x} = \frac{\partial c}{\partial t} + f'(c(x(t),t))\frac{\partial c}{\partial x} = 0$$

- **Value of the solution at $(x(t),t)$ depends only on initial value:** $c(x(t),t) = c_0(x_0)$

Characteristic curves are lines

$$\frac{dx}{dt} = f'(c(x(t),t)) = f'(c_0(x_0)) \quad \textbf{implies} \quad x(t) = x_0 + f'(c_0(x_0))t$$

Method of characteristics to compute solution:

- **For each (x,t) find x_0 such that $x = x_0 + f'(c_0(x_0))t$ (amounts to solve a nonlinear equation)**
- **For those x, t, x_0, one has $c(x,t) = c(x_0 + f'(c_0(x_0))t) = c_0(x_0)$**
- **Propagation of information at speed $f'(c_0(x_0))$**
- **For fixed t, define the origin of the characteristic line $x_0(x,t)$ as a function of the point x it will cross at t**

Origin of the characteristic line (1)

$$x = x_0(x,t) + f'(c_0(x_0(x,t)))t$$

To compute derivatives of $x_0(x,t)$ as function of x derive both sides:

$$1 = \frac{\partial x_0(x,t)}{\partial x} + \frac{\partial}{\partial x}\left(f'(c_0(x_0(x)))t\right)$$
$$1 = \frac{\partial x_0(x,t)}{\partial x} + f''(c_0(x_0(x)))c_0'(x_0(x))t\frac{\partial x_0(x,t)}{\partial x}$$
$$\frac{\partial x_0(x,t)}{\partial x} = \frac{1}{1 + f''(c_0(x_0))c_0'(x_0(x))t}$$

Derivative becomes infinite for

$$t \approx -\frac{1}{f''(c_0(x_0))c_0'(x_0)}$$

Origin of the characteristic line (2)

$$x = x_0(x,t) + f'(c_0(x_0(x,t)))t$$

To compute derivatives of $x_0(x,t)$ as function of t, derive both sides:

$$0 = \frac{\partial x_0(x,t)}{\partial t} + \frac{\partial}{\partial t}\left(f'(c_0(x_0(x)))t\right)$$
$$0 = \frac{\partial x_0(x,t)}{\partial t} + f''(c_0(x_0(x)))c_0'(x_0(x))t\frac{\partial x_0(x,t)}{\partial t} + f'(c_0(x_0(x)))$$
$$\frac{\partial x_0(x,t)}{\partial t} = -\frac{f'(c_0(x_0(x)))}{1 + f''(c_0(x_0))c_0'(x_0(x))t}$$

Derivative becomes infinite for

$$t \approx -\frac{1}{f''(c_0(x_0))c_0'(x_0)}$$

Development of singularities (1)

Expect problems if

$$f''(c_0(x_0)) < 0 \quad c_0'(x_0) > 0 \quad \text{or} \quad f''(c_0(x_0)) > 0 \quad c_0'(x_0) < 0$$

- **Let $x_1 < x_2$ and $f'(c_0(x_1)) > f'(c_0(x_2))$: then at some point the characteristics starting from x_1, x_2 will meet**

$$x_1(t) = x_1 + f'(c_0(x_1))t = x_2(t) = x_2 + f'(c_0(x_2))t$$

- **Bad news: uniqueness of characteristics breaks down**

Development of singularities (2)

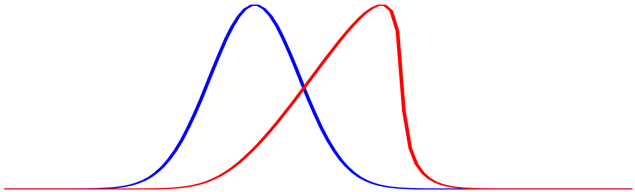
Compute derivatives of solution with respect to space and time:

$$\frac{\partial c}{\partial x}(x,t) = \frac{\partial}{\partial x} c_0(x_0) = c_0'(x_0)\frac{\partial x_0}{\partial x}$$
$$\frac{\partial c}{\partial t}(x,t) = \frac{\partial}{\partial t} c_0(x_0) = c_0'(x_0)\frac{\partial x_0}{\partial t}$$

Derivative becomes infinite for

$$t \approx -\frac{1}{f''(c_0(x_0))c_0'(x_0)}$$

Let's have a look



- **Burgers equation: development of a singularity from regular initial data**
- **Solution is computed by the method of characteristics**

Weak solutions

The function $c(x,s)$ is a weak solution of the nonlinear conservation law for $s \in [0,t]$ with initial data $c_0(x)$ and boundary data $c(0,s) = g_0(s),\ c(L,s) = g_L(s)$ if for any $\phi \in \mathcal{C}^\infty([0,L] \times [0,t])$ it holds

$$
\begin{aligned}
&\int_0^L c(x,t)\phi(x,t)\ dx - \int_0^L c_0(x)\phi(x,0)\ dx \\
&+ \int_0^t f(g_L(s))\phi(L,s)\ ds - \int_0^t f(g_0(s))\phi(0,s)\ ds \\
&- \int_0^t \int_0^L \left[c\frac{\partial \phi}{\partial s} + f(c)\frac{\partial \phi}{\partial x} \right]\ dx\ ds = 0
\end{aligned}
$$

Also in this case, the definition extends the classical concept of solution: proof that classical solutions are also weak solutions is analogous to that of the advection case

Weak solutions as viscosity solutions

Nonlinear conservation law with viscosity

$$\frac{\partial c_\epsilon}{\partial t} + \frac{\partial}{\partial x} f(c_\epsilon) = \epsilon \frac{\partial^2 c_\epsilon}{\partial x^2}$$

- Due to the parabolic term, solutions of nonlinear conservation laws with viscosity are always differentiable (classical solutions)
- Weak solutions can be obtained as vanishing viscosity limit

$$c(x,t) = \lim_{\epsilon \to 0} c_\epsilon(x,t)$$

- Unpractical to do it numerically all the time: direct numerical approximation of weak solutions is essential

The Riemann problem

A special initial and boundary value problem

Riemann problem

Consider the nonlinear conservation law for $x \in [0, L]$
with initial condition

$$c_0(x) = c_{left} \quad x \leq x_0 \qquad c_0(x) = c_{right} \quad x > x_0 \qquad x \in (0, L)$$

and boundary conditions $c(0, t) = c_{left}, \ c(L, t) = c_{right}.$

- Riemann problem from Bernhard Riemann (1826-1866), great German mathematician, founder of modern differential geometry
- Exact solution can be computed in many cases: useful to understand physics and to build or test numerical methods

Riemann problem: shock wave solutions (1)

Look for shock waves: weak solutions in which initial jump propagates at speed s

- **Assume that the solution has the form**

$$c(x,t) = c_{left} \quad x \leq x(t) = x_0 + st \qquad c(x,t) = c_{right} \quad x > x(t) = x_0 + st$$

- **If $c(x,t)$ is a weak solution, it must satisfy the definition also with $\phi = 1$ over interval $[0, L]$, therefore**

$$\int_0^L c(x,t)\ dx - \int_0^L c_0(x)\ dx + \int_0^t f(c_{right})\ ds - \int_0^t f(c_{left})\ ds = 0$$

Riemann problem: shock wave solutions (2)

- **Using the assumptions on initial data and solution shape, one should have**

$$\int_0^{x(t)} c_{left}\ dx + \int_{x(t)}^L c_{right}\ dx - \int_0^L c_0(x)\ dx$$
$$+ \int_0^t f(c_{right})\ ds - \int_0^t f(c_{left})\ ds = 0$$

- **Taking time derivative of both sides one obtains**

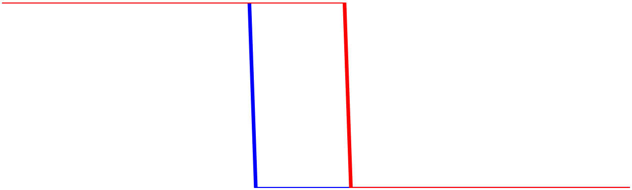
$$x'(t)c_{left} - x'(t)c_{right} + f(c_{right}) - f(c_{left}) = 0$$

- **A weak solution can be a shock wave solution only if**

$$x'(t) = s = \frac{f(c_{right}) - f(c_{left})}{c_{right} - c_{left}}$$

- **Rankine-Hugoniot jump condition**

Let's have a look



- **A shock wave solution for the Burgers equation**
- **Shock wave speed is totally determined by initial data**
- **Is it the only possibility? if $c_{left} < c_{right}$ and $f'(c_{right}) > f'(c_{left})$ velocity on the right side should be larger than that on the left, leaving a gap in between**

Riemann problem: unphysical shock wave solutions

Suppose $c_{left} < c_{right}$ **and** $f'(c_{right}) > f'(c_{left})$:

- **Mathematically, it is still possible to have shock wave solution with speed**

$$s = \frac{f(c_{right}) - f(c_{left})}{c_{right} - c_{left}}$$

- **Physically, this is incorrect: shocks are correct solutions only if**

$$f'(c_{right}) < s < f'(c_{left})$$

- **Lax entropy condition: avoids non uniqueness of weak solution**

Riemann problem: rarefaction wave solutions (1)

Look for rarefaction waves: continuous weak solutions constant along lines passing through $x0$ at $t=0$

- **Assume that solution is self similar:** $c(x,t) = \bar{c}((x-x0)/t)$
- **Compute derivatives**

$$\frac{\partial c}{\partial t}(x,t) = -\frac{x-x0}{t^2}\bar{c}'\left(\frac{x-x0}{t}\right) \qquad \frac{\partial c}{\partial x}(x,t) = \frac{x-x0}{t}\bar{c}\left(\frac{x-x0}{t}\right)$$

- **Impose condition that conservation law is satisfied, using the fact that**

$$\frac{\partial}{\partial x}f(c) = f'(c)\frac{\partial c}{\partial x}$$

Riemann problem: rarefaction wave solutions (2)

- **Resulting condition is**

$$f'\left(\bar{c}\left(\frac{x-x0}{t}\right)\right)=\frac{x-x0}{t}$$

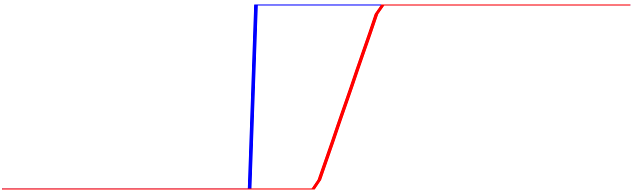
- **If f' is invertible and $g(f'(u))=u,$ then**

$$c(x,t)=\bar{c}\left(\frac{x-x0}{t}\right)=g\left(\frac{x-x0}{t}\right)$$

- **Appropriate solution of Riemann problem for x,t such that**

$$f'(c_{left})\leq\frac{x-x0}{t}\leq f'(c_{right})$$

Let's have a look



- A rarefaction wave solution for the Burgers equation
- The rate at which the rarefaction fan widens is totally determined by initial data

Summary of the first lecture

- The advection equation as the prototype hyperbolic problem: waves propagating at finite speed
- Classical (differentiable) and weak (non differentiable) solutions
- Nonlinear conservation laws: more complex waves and breakdown of regularity
- The Riemann problem: shocks and rarefactions