Introduction to finite volume methods for hyperbolic equations

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Outline of the lecture

Integral form of conservation laws and FV methods

Numerical fluxes

Reconstruction and monotonization procedures

FV methods for hyperbolic systems

Key concepts introduced in the fourth lecture

- **General derivation of a finite volume (FV) method for a nonlinear conservation law**
- **Numerical fluxes for scalar conservation laws**
- **Reconstruction techniques and monotonization procedures**
- **Finite volume approximations of hyperbolic systems**

Integral form of conservation laws and FV methods

Integral form of conservation laws

- **Start from general nonlinear conservation law**

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} f(c) = 0$$

- **Derive equation for $\int_a^b c \, dx$ integrating over interval (a, b)**

$$\frac{d}{dt} \int_a^b c(x,t) \, dx = -\left[f(c(b,t)) - f(c(a,t)) \right]$$

- **Define average $\bar{c}(t)$ over $(a, b,)$ derive its evolution equation**

$$\bar{c}(t) = \frac{1}{(b-a)} \int_a^b c(x,t) \, dx$$
$$\frac{d}{dt} \bar{c}(t) = -\frac{1}{(b-a)} \left[f(c(b,t)) - f(c(a,t)) \right]$$

Finite volume methods (1)

- **Divide solution interval $[0, L]$ in non overlapping intervals $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ $\quad i = 1, \dots, N$ of size $\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ called control volumes**

$$[0, L] = \bigcup_{i=1}^{N} [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$$

- **Introduce discrete variables**

$$c_i(t) \approx \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} c(x, t)\ dx \qquad f_{i\pm\frac{1}{2}}(t) \approx f(c(x_{i\pm\frac{1}{2}}, t))$$

- **Assume that $c_i(t)$ satisfy equations**

$$\frac{d}{dt} c_i(t) = -\frac{1}{\Delta x_i} \left[f_{i+\frac{1}{2}}(t) - f_{i-\frac{1}{2}}(t) \right]$$

Finite volume methods (2)

- $f_{i\pm\frac{1}{2}}(t)$ **numerical fluxes: key to the definition of the method**
- **In general, introduce function $\hat{f}(a,b)$ also called numerical flux, such that**

$$f_{i\pm\frac{1}{2}}(t) = \hat{f}(c^-_{i\pm\frac{1}{2}}(t), c^+_{i\pm\frac{1}{2}}(t))$$

- $c^-_{i\pm\frac{1}{2}}(t), c^+_{i\pm\frac{1}{2}}(t)$: **reconstructions of solution at $x_{i\pm\frac{1}{2}}$ using information from the left (-) and/or the right side (+)**
- **FV methods are conservative at discrete level**

$$\frac{d}{dt}\sum_{i=1}^{N} c_i(t)\Delta x_i = -\sum_{i=1}^{N}\left[f_{i+\frac{1}{2}}(t) - f_{i-\frac{1}{2}}(t)\right] = -\left[f_{N+\frac{1}{2}}(t) - f_{\frac{1}{2}}(t)\right]$$

Numerical fluxes

General properties of numerical fluxes

- **Consistency of the numerical flux with the conservation law:**

$$f(u) = \hat{f}(u, u)$$

- **Lipschitz continuity of the numerical flux:**

$$|\hat{f}(a_1, b) - \hat{f}(a_2, b)| \leq K|a_1 - a_2| \qquad |\hat{f}(a, b_1) - \hat{f}(a, b_2)| \leq K|b_1 - b_2|$$

- **Monotonicity of numerical flux: $\hat{f}(a, b)$ is non decreasing function of a and non increasing function of b**

General properties of resulting FV methods (1)

- **Let $c_i^{ex}(t), i = 1, \dots, N$ be the control volume averages of the values of exact solution, $\Delta x = \max_{i=1,\dots,N} \Delta x_i$**
- **Spatial local truncation error:**

$$\tau_i(\Delta x) = \left| \frac{d}{dt} c_i^{ex}(t) \right.$$
$$\left. + \frac{1}{\Delta x_i} \left[\hat{f}(c_{i+\frac{1}{2}}^{ex,-}(t), c_{i+\frac{1}{2}}^{ex,+}(t)) - \hat{f}(c_{i-\frac{1}{2}}^{ex,-}(t), c_{i-\frac{1}{2}}^{ex,+}(t)) \right] \right|$$

- **Spatial discretization is consistent of order p if for sufficiently small Δx, uniformly in $t \in [0, T]$**

$$\max_{i=1,\dots,N} |\tau_i(\Delta x)| \leq K \Delta x^p$$

General properties of resulting FV methods (2)

- Spatial discretization is convergent of order p if for sufficiently small Δx, uniformly in $t \in [0, T]$

$$\max_{i=1,\dots,N} |c_i^{ex}(t) - c_i(t)| \leq K \Delta x^p$$

- Consistency is a necessary condition for convergence, but for simplicity often it is the only one that is checked
- In general, the order of accuracy of the reconstruction method employed determines the order of consistency and convergence of the FV method
- Total accuracy of numerical solutions depends on space and time discretizations: order of convergence of space and time method should be related

Some general remarks on FV methods

- **In some presentations, it is taken for granted that time discretization method is forward Euler method:**

$$c_i^{n+1} = c_i^n - \frac{\Delta t}{\Delta x_i}\left[\hat{f}(c_{i+\frac{1}{2}}^{n,-}, c_{i+\frac{1}{2}}^{n,+}) - \hat{f}(c_{i-\frac{1}{2}}^{n,-}, c_{i-\frac{1}{2}}^{n,+})\right]$$

- **Some fluxes encountered in the literature are only defined coupled to the forward Euler method**
- **Midpoint formula: average over interval and midpoint value are equivalent up to second order error**

$$\bar{c}(t) = \frac{1}{(b-a)}\int_a^b c(x,t)\ dx = c\left(\frac{a+b}{2}\right) + O\left((b-a)^2\right)$$

- **Consequence: finite volume methods of convergence order $p \leq 2$ can be reinterpreted as methods to compute approximations of midpoint values**

Lax Friedrichs flux

$$\hat{f}^{LF}(a,b) = \frac{f(b)+f(a)}{2} - \frac{\alpha}{2}(b-a) \qquad \alpha = \max|f'(c)|$$

- **Lipschitz and monotonic flux**
- **Local Lax Friedrichs or Rusanov variant:**

$$\alpha = \max\{|f'(c)| \ \text{ on some local set of states}\}$$

- **Linear advection: both revert to upwind flux**

Centered flux

$$\hat{f}^C(a,b) = \frac{f(b)+f(a)}{2}$$

- **Lipschitz but non monotonic flux, the extra term in Lax Friedrichs adds numerical diffusion and yields monotonicity:**

$$\begin{aligned}
&\hat{f}^{LF}(c_i, c_{i+1}) - \hat{f}^{LF}(c_{i-1}, c_i) \\
&= \hat{f}^C(c_i, c_{i+1}) - \hat{f}^C(c_{i-1}, c_i) - \frac{\alpha}{2}(c_{i+1} - c_i) + \frac{\alpha}{2}(c_i - c_{i-1}) \\
&= \frac{f(c_{i+1}) - f(c_{i-1})}{2} - \frac{\alpha}{2}(c_{i+1} - 2c_i + c_{i-1})
\end{aligned}$$

- **Must be modified with cell size/edge dependent weighting in the case of non uniform meshes**
$\hat{f}^C(c^-_{i+\frac{1}{2}}, c^+_{i+\frac{1}{2}}) = (f(c^-_{i+\frac{1}{2}})\Delta x_{i+1} + f(c^+_{i+\frac{1}{2}})\Delta x_i)/(\Delta x_{i+1} + \Delta x_i)$

Roe flux

$$\hat{f}^R(a,b) = \sigma^-(\alpha)f(b) + \sigma^+(\alpha)f(a)$$

$$\alpha = \frac{f(b)-f(a)}{b-a} \qquad \sigma^\pm(\alpha) = \frac{1 \pm \mathrm{sign}(\alpha)}{2}$$

- **Lipschitz and monotonic flux**
- **Does not identify unphysical shocks: entropy fixes necessary**
- **Linear advection: reverts to upwind flux**

Godunov flux

$$\begin{aligned}\hat{f}^G(a,b) &= \sigma^+(\alpha)\min\{f(c) : c \in [a,b]\} \\ &+ \sigma^-(\alpha)\min\{f(c) : c \in [b,a]\}\end{aligned}$$

$$\alpha = b - a \qquad \sigma^{\pm}(\alpha) = \frac{1 \pm \mathrm{sign}(\alpha)}{2}$$

- **Lipschitz and monotonic flux**
- **For piecewise constant reconstructions, it can be proven that**

$$\hat{f}^G(c_i, c_{i+1}) = f(c_{i+\frac{1}{2}}^{Riemann}),$$

where $c_{i+\frac{1}{2}}^{Riemann}$ is the exact solution of Riemann problem with $x_0 = x_{i+\frac{1}{2}}$ and data c_i, c_{i+1}

- **Linear advection: reverts to upwind flux**

Reconstruction and monotonization procedures

Need for reconstruction techniques

- **Average and midpoint value coincide up to a Δx^2 error**
- **Point value at $x_{i+\frac{1}{2}}$ is needed to compute numerical flux**
- **Approximation with average values c_i, c_{i+1} : usually first order consistent in space, second order only in some special cases like centered flux or Lax-Wendroff flux**

$$\hat{f}^{LW}(c_i, c_{i+1}) = \frac{f(c_{i+1}) + f(c_i)}{2} - \alpha\frac{\Delta t}{\Delta x}(f(c_{i+1}) - f(c_i))$$

$$\alpha = f'\left(\frac{c_{i+1} + c_i}{2}\right)$$

- **Approximation with piecewise linear reconstruction: define approximation of slope σ_i and set**

$$c(x_{i+\frac{1}{2}}) \approx c_i + \sigma_i\frac{\Delta x_i}{2}$$

Higher order reconstruction techniques

- **For each volume, let** $f \in \mathcal{C}^n$ **and** $f_i = \int_{i-\frac{1}{2}}^{i+\frac{1}{2}} f(x)\ dx$
- **For reconstruction of order of accuracy** m, **use reconstruction formula** $R_i(x) = \sum_{j=i-k}^{i+l} \alpha_j f_j$ **such that**

$$\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} R_i(x)\ dx = \Delta x_j f_j \qquad j = i-k, \dots, i+l$$

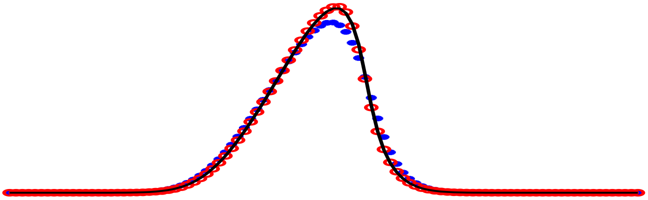
$$f(x) = R_i(x) + O(\Delta x^m) \quad x \in \left[i - \frac{1}{2}, i + \frac{1}{2}\right]$$

- **Examples of third order accurate reconstructions (on uniform mesh):**

$$R_i^{(1)}(x_{i+\frac{1}{2}}) = \frac{1}{3} f_{i-2} - \frac{7}{6} f_{i-1} + \frac{11}{6} f_i \quad R_i^{(2)}(x_{i+\frac{1}{2}}) = -\frac{1}{6} f_{i-1} + \frac{5}{6} f_i - \frac{1}{3} f_{i+1}$$

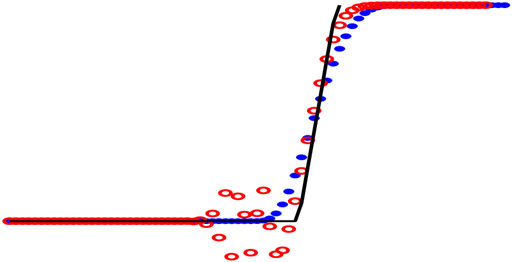
- **All methods of order higher than one everywhere create spurious oscillations**

Let's have a look: smooth solutions



- **Rusanov flux, piecewise constant reconstruction, forward Euler method in time (first order)**
- **Rusanov flux, piecewise linear reconstruction, Heun method in time (second order)**
- **Reduction in numerical diffusion if full second order method in space and time is used**

Let's have a look: less smooth solutions



- **Rusanov flux, piecewise constant reconstruction, forward Euler method in time (first order)**
- **Rusanov flux, piecewise linear reconstruction, Heun method in time (second order)**
- **Spurious oscillations if second order method is used**

Monotonicity concepts

Monotone method

$c_i^{n+1} = G(c_{i-k}^n, \cdots, c_{i+l}^n)$ with G non decreasing function of its arguments

Total variation diminishing (TVD) method

$$\sum_i |c_{i+1}^{n+1} - c_i^{n+1}| \leq \sum_i |c_{i+1}^n - c_i^n|$$

- In general, monotone implies TVD
- For linear methods (such that $c_i^{n+1} = \sum_{j=i-k}^{i+l} \alpha_j c_j$, for linear advection), TVD implies monotone
- Using monotone flux is necessary condition for monotonicity

Paradoxes of monotonicity

- A linear method that is monotone is at most first order accurate (Godunov's theorem 1959)
- As a result, a TVD method can be more than first order accurate only if it is nonlinear (even for linear equations!)
- A TVD method in $d > 1$ dimensions is at most first order accurate (Goodman and Leveque theorem, 1985)
- In practice, TVD methods exist that are more than first order accurate on most of the domain in 1D
- These methods can be extended to $d > 1$ dimensions

Flux limiting

- Nonlinear blending of first order, monotone flux and high order, non monotone flux

$$\hat{f}_{i+\frac{1}{2}} = \left(1 - \phi_{i+\frac{1}{2}}\right) \hat{f}^{M}(c_i^n, c_{i+1}^n) + \phi_{i+\frac{1}{2}} \hat{f}^{NM}(c_{i+\frac{1}{2}}^{n,-}, c_{i+\frac{1}{2}}^{n,+})$$

- Flux limiter function $\phi_{i+\frac{1}{2}}$ depends non linearly on solution values $c_{i-k}^n, \ldots, c_{i+l}^n$
- $\phi_{i+\frac{1}{2}} \approx 0$ close to discontinuities: monotone first order flux is used
- $\phi_{i+\frac{1}{2}} \approx 1$ when solution is smooth: non monotone higher order flux is used

Slope limiting

- **Approximation with piecewise linear reconstruction: define approximation of slope σ_i and set**

$$c(x_{i+\frac{1}{2}}) \approx c_i + \sigma_i \frac{\Delta x_i}{2}$$

- **MUSCL (monotone, upstream centered scheme for conservation laws) approach by Van Leer: limit the slope σ_i so that too large oscillations cannot take place**

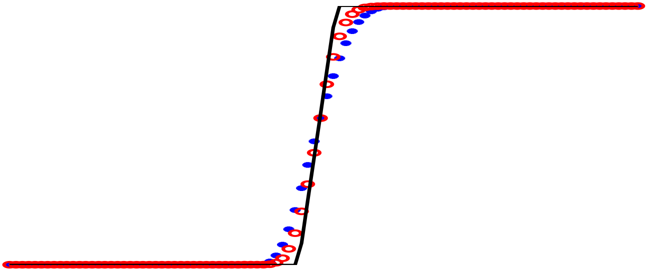
$$\sigma_i = \psi \left(\frac{c_i - c_{i-1}}{\Delta x_{i-\frac{1}{2}}}, \frac{c_{i+1} - c_i}{\Delta x_{i+\frac{1}{2}}} \right)$$

- **Common slope limiters: minmod function, superbee limiter, MC limiter**
- **Slope limiters can be rewritten as flux limiters**

Essentially non oscillatory (ENO) methods

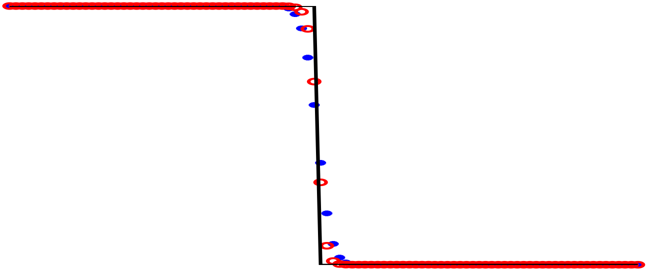
- To reach accuracy orders higher than 2 with minimum amount of oscillations, build hierarchically high order reconstructions (divided differences representation of interpolation polynomial)
- Compare different possible reconstructions of the same degree and choose for each degree the one that gives less oscillations

Let's have a look: less smooth solutions



- **Rusanov flux, piecewise constant reconstruction in space, forward Euler method in time**
- **Rusanov flux, piecewise linear reconstruction with slope limiter in space, Heun method in time**
- **Slope limiter removes spurious oscillations but maintains second order accuracy**

Let's have a look: shock solutions



- **Rusanov flux, piecewise constant reconstruction in space, forward Euler method in times**
- **Rusanov flux, piecewise linear reconstruction with slope limiter in space, Heun method in time**
- **Slope limiter removes spurious oscillations but close to shocks accuracy is essentially first order**

FV methods for hyperbolic systems

Nonlinear hyperbolic systems

$$\frac{\partial \mathbf{c}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{c}) = \mathbf{0}$$

- **Mathematical definition: system is hyperbolic if $\mathbf{f}'(\mathbf{c})$ is diagonalizable and has real eigenvalues, strictly hyperbolic if it has real and distinct eigenvalues $\lambda_i(\mathbf{c})$**

$$\mathbf{f}'(\mathbf{c})\mathbf{t}_i(\mathbf{c}) = \lambda_i(\mathbf{c})\mathbf{t}_i(\mathbf{c}) \quad i = 1, \ldots, d$$

- **Decomposition of Jacobian matrix**

$$\mathbf{f}'(\mathbf{c}) = \mathbf{T}(\mathbf{c})\mathbf{\Lambda}\mathbf{T}^{-1}(\mathbf{c}) \qquad \mathbf{\Lambda} = \mathrm{diag}(\lambda_i(\mathbf{c}))$$

FV methods for hyperbolic systems

$$\frac{d}{dt}\mathbf{c}_i(t) = -\frac{1}{\Delta x_i}\left[\hat{\mathbf{f}}(c^-_{i+\frac{1}{2}}(t), c^+_{i+\frac{1}{2}}(t)) - \hat{\mathbf{f}}(c^-_{i-\frac{1}{2}}(t), c^+_{i-\frac{1}{2}}(t))\right]$$

- **Vector flux function $\hat{\mathbf{f}}$must be introduced**
- **Centered, Lax Friedrichs, Rusanov fluxes can be generalized to the case of systems: for LF, R, set**

$$\alpha = \max\{\max|\lambda_k(\mathbf{c})|, \quad k = 1, \dots, d\}$$

- **For the generalization of Godunov method, the exact solution of the Riemann problem must be known**

The Godunov method

$$\hat{\mathbf{f}}^G(\mathbf{c}_i, \mathbf{c}_{i+1}) = \mathbf{f}\left(\mathbf{c}_{i+\frac{1}{2}}^{Riemann}\right)$$

- $\mathbf{c}_{i+\frac{1}{2}}^{Riemann}$: **exact solution of Riemann problem with $x_0 = x_{i+\frac{1}{2}}$ and data $\mathbf{c}_i, \mathbf{c}_{i+1}$**
- **Available for the Euler equations and other selected problems, difficult to compute in general**
- **Exact Riemann solver is expensive: in practice approximate Riemann solvers are mostly used**

Main families of approximate Riemann solvers

- **Flux difference splitting: define a matrix $\mathbf{A} = \mathbf{A}(\mathbf{a}, \mathbf{b})$ such that**

$$\hat{\mathbf{f}}(\mathbf{a}, \mathbf{b}) = \frac{\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{b})}{2} - \frac{\mathbf{A}}{2}(\mathbf{b} - \mathbf{a}) + o(\|\mathbf{b} - \mathbf{a}\|)$$

- **Flux vector splitting: define numerical flux as**

$$\hat{\mathbf{f}}(\mathbf{a}, \mathbf{b}) = \mathbf{f}^{+}(\mathbf{a}) + \mathbf{f}^{-}(\mathbf{b})$$

where positive and negative parts $\mathbf{f}^{\pm}$ of the flux vector function have positive and negative eigenvalues, respectively

Flux difference splitting: the Roe method (1)

Find Roe matrix $\mathbf{A} = \mathbf{A}(\mathbf{a}, \mathbf{b})$ such that

- Consistency: $\mathbf{A}(\mathbf{a}, \mathbf{a}) = \mathbf{f}'(\mathbf{a})$
- Hyperbolicity: for all $\mathbf{A}(\mathbf{a}, \mathbf{b})$ is diagonalizable with real eigenvalues
- Generalized RH condition: $\mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a}) = \mathbf{A}(\mathbf{a}, \mathbf{b})(\mathbf{b} - \mathbf{a})$
- Roe matrix computed for the Euler equations and a number of other hyperbolic systems, corrections necessary to avoid unphysical shock solutions at rarefaction waves

Flux Vector Splitting approaches: Rusanov flux

- **Define** $\bar{\lambda}_i = \max |\lambda_i(\mathbf{c})|$, $\bar{\mathbf{\Lambda}} = \mathrm{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_d)$
- **Decompose flux as**

$$\mathbf{f}(\mathbf{c})^{\pm} = \frac{1}{2}\left[\mathbf{f}(\mathbf{c}) + \mathbf{T}(\mathbf{c})\bar{\mathbf{\Lambda}}\mathbf{T}^{-1}(\mathbf{c})\right]$$

- **Can be combined with MUSCL-like monotonization approaches**
- **Numerical diffusion affects contact discontinuities**

Flux Vector Splitting approaches: Steger-Warming flux

- For homogeneous systems $\mathbf{f}(\mathbf{c}) = \mathbf{f}'(\mathbf{c})\mathbf{c}$ (Euler equations)
- Define

$$\mathbf{f}'(\mathbf{c})^{\pm} = \mathbf{T}\boldsymbol{\Lambda}^{\pm}\mathbf{T}^{-1} \qquad \boldsymbol{\Lambda}^{\pm} = \mathrm{diag}(\lambda_i^{\pm}(\mathbf{c}))$$
$$a^{+} = \max(a, 0) \qquad a^{-} = \min(a, 0)$$

- Can be combined with MUSCL-like monotonization approaches
- Numerical diffusion affects contact discontinuities

Finite Volume methods in multiple dimensions (1)

- **Nonlinear conservation law**

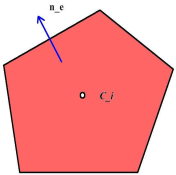
$$\frac{\partial c}{\partial t} + \frac{\partial f(c)}{\partial x} + \frac{\partial g(c)}{\partial y} = 0$$

- **Let $\bar{c}_i$ be the average over and arbitrary volume Ω_i. Integrate and apply Green's theorem:**

$$\frac{d}{dt}\bar{c}_i + \int_{\partial\Omega_i} \left(f(c)n_x + g(c)n_y\right) \, ds = 0$$

- **Approximate numerically the boundary integral using one dimensional numerical fluxes along the direction normal to each volume edge**

Finite Volume methods in multiple dimensions (2)



- Fully discrete (in space) multidimensional FV method:

$$|\Omega|_i \frac{d}{dt} c_i + \sum_{e \in \partial\Omega_i} \hat{f}(c_e^-(t), c_e^+(t)) = 0$$

- Unstructured meshes with arbitrary polygonal cells can be employed
- High order versions: easy on Cartesian meshes, more complex on arbitrary meshes, require high order quadrature rules for computation of numerical fluxes

Summary of the fourth lecture

- **Finite volume** methods: naturally conservative method to compute approximation of solution **averages**
- Key ingredients: numerical **fluxes** and **reconstruction** techniques
- Finite volume approximations of **hyperbolic** equations and systems: no systematic error in the prediction of **shock wave speeds**
- Easy extension to multidimensional case for **unstructured meshes**