Introduction to discontinuous finite element methods for hyperbolic equations, part (2)

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Keio University, 9.16.2015

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Outline of the lecture

Matrix form of discrete equations for DG method

Monotonization of DG discretizations

Examples of time discretizations

Degree adaptivity

Numerical results

Suggested reading

Key concepts introduced in the sixth lecture

- Matrix form of discrete equations for DG method
- Monotonization of DG discretizations
- Examples of time discretizations
- An algorithm for degree adaptivity
- Numerical results
- Suggested reading

Matrix form of discrete equations for DG method

DG methods, fully discrete in space

For all $k = 1, \ldots, N_e,$ **for all basis functions** $\phi_{k,i}(x), i = 0, \ldots, r(k)$

$$\sum_{j=1}^{N(k)} c'_{k,j}(t) m_{i,j} = -\hat{f}(c^-_{h,k+\frac{1}{2}}(t), c^+_{h,k+\frac{1}{2}}(t))\phi_{k,i}(x_{k+\frac{1}{2}})$$
$$+\hat{f}(c^-_{h,k-\frac{1}{2}}(t), c^+_{h,k-\frac{1}{2}}(t))\phi_{k,i}(x_{k-\frac{1}{2}}) + F_{k,i}(c_h(x,t))$$

$$m_{i,j} = \sum_{l=1}^{q} \phi_{k,j}(x_l^{(k)})\phi_{k,i}(x_l^{(k)}) w_l$$

$$F_{k,i}(c_h(x,t)) = \sum_{l=1}^{q} f(c_h(x_l^{(k)}, t)) \frac{\partial \phi_{k,i}}{\partial x}(x_l^{(k)}) w_l$$

where $x_l^{(k)}, l = 1, \ldots, q$ **denote the Gaussian nodes mapped onto** $x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}$ **from** $[-1, 1]$

Discrete DG methods, matrix form (1)

$$\mathbf{c}_k(t) = \begin{bmatrix} c_{k,0}(t) \\ \cdots \\ \cdots \\ c_{k,r(k)}(t) \end{bmatrix} \qquad \mathbf{M}_k = \begin{bmatrix} m_{0,0} & \cdots & m_{0,r(k)} \\ m_{1,0} & \cdots & m_{1,r(k)} \\ \cdots & \cdots & \cdots \\ m_{r(k),0} & \cdots & m_{r(k),r(k)} \end{bmatrix}$$

Lagrange basis: full mass matrix $\mathbf{M}_k$, Legendre basis: diagonal mass matrix $\mathbf{M}_k$

Discrete DG methods, matrix form (2)

$$\mathbf{F}(\mathbf{c})_k(t) =$$

$$\begin{bmatrix} -\hat{f}_{k+\frac{1}{2}}\phi^{+}_{k,0} + \hat{f}_{k-\frac{1}{2}}\phi^{-}_{k,0} + \sum_{l=1}^{q} f(c_h(x_l^{(k)},t))\phi'_{k,0}(x_l^{(k)})w_l \\ \cdots \\ \cdots \\ -\hat{f}_{k+\frac{1}{2}}\phi^{+}_{k,r(k)} + \hat{f}_{k-\frac{1}{2}}\phi^{-}_{k,r(k)} + \sum_{l=1}^{q} f(c_h(x_l^{(k)},t))\phi'_{k,r(k)}(x_l^{(k)})w_l \end{bmatrix}$$

Final matrix form of the equation

$$\mathbf{M}_k \frac{d\mathbf{c}_k}{dt} = \mathbf{F}(\mathbf{c})_k = \mathbf{F}(\mathbf{c}_{k-1}, \mathbf{c}_k, \mathbf{c}_{k+1})_k$$

Discrete DG methods, P1 case (1)

$$\mathbf{c}_k(t) = \begin{bmatrix} c_{k,0}(t) \\ c_{k,1}(t) \end{bmatrix} \qquad \mathbf{M}_k = \begin{bmatrix} m_{0,0} & m_{0,1} \\ m_{1,0} & m_{1,1} \end{bmatrix}$$

$$\mathbf{F}(\mathbf{c})_k(t) = \begin{bmatrix} -\hat{f}_{k+\frac{1}{2}}\phi^+_{k,0} + \hat{f}_{k-\frac{1}{2}}\phi^-_{k,0} + \sum_{l=1}^q f(c_h(x_l,t))\phi'_{k,0}(x_l^{(k)})w_l \\ -\hat{f}_{k+\frac{1}{2}}\phi^+_{k,1} + \hat{f}_{k-\frac{1}{2}}\phi^-_{k,1} + \sum_{l=1}^q f(c_h(x_l,t))\phi'_{k,1}(x_l^{(k)})w_l \end{bmatrix}$$

$c_{k,0}(t)$ **represents average of the solution over element** k
$c_{k,1}(t)$ **represents slope of the solution over element** k
Simplifications: $\phi'_{k,0}(x_l^{(k)}) = 0,\ q = 2$ **is sufficient**

Discrete DG methods, P1 case (2)

Employ Legendre basis: $m_{1,0} = m_{0,1} = 0$; **employ two point Gauss Legendre formula:** $q = 2$

$$\frac{dc_{k,0}}{dt} = -\frac{1}{m_{0,0}} \left(\hat{f}_{k+\frac{1}{2}} \phi^{+}_{k,0} - \hat{f}_{k-\frac{1}{2}} \phi^{-}_{k,0} \right)$$

$$\frac{dc_{k,1}}{dt} = -\frac{1}{m_{1,1}} \left(\hat{f}_{k+\frac{1}{2}} \phi^{+}_{k,1} - \hat{f}_{k-\frac{1}{2}} \phi^{-}_{k,1} \right) + \frac{1}{m_{1,1}} \sum_{l=1}^{2} f(c_h(x_l^{(k)}, t)) \phi'_{k,1}(x_l^{(k)}) w_l$$

Equation for $c_{k,0}(t)$ is entirely equivalent to finite volume discretization, equation for $c_{k,1}(t)$ allows to have full second order accuracy on smooth solutions solutions

Monotonization of DG discretizations

Slope limiting approaches

- Most common technique, first proposed by Cockburn and Shu (see references at the end of the lecture)
- Use standard slope limiting methods to avoid spurious oscillations
- In the framework of P1 discretizations: modify slope value after each step of time discretization has been performed

$$c_{k,1}^{\mathrm{lim}} = \mathrm{minmod}\left(c_{k,1}, \frac{c_{k+1,0} - c_{k,0}}{\Delta x}, \frac{c_{k,0} - c_{k-1,0}}{\Delta x} \right)$$

- Extensions to higher order are available, but application of this limiter makes the method locally first order accurate only

Flux limiting approaches (1)

- **Equation for the average identical to finite volume method**

$$\frac{dc_{k,0}}{dt} = -\frac{1}{m_{0,0}} \left(\hat{f}_{k+\frac{1}{2}} \phi^{+}_{k,0} - \hat{f}_{k-\frac{1}{2}} \phi^{-}_{k,0} \right)$$

- **Use standard flux limiting technique as in finite volumes**

$$\hat{f}_{k+\frac{1}{2}} = \hat{f}^{L}_{k+\frac{1}{2}} + C^{\lim}_{k+\frac{1}{2}} A_{k+\frac{1}{2}} \qquad A_{k+\frac{1}{2}} = (\hat{f}^{H}_{k+\frac{1}{2}} - \hat{f}^{L}_{k+\frac{1}{2}}) \qquad C^{\lim}_{k+\frac{1}{2}} \in [0,1]$$

- **In principle, any standard flux limiter can be employed**

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Flux limiting approaches (2)

Flux corrected transport, well known monotonization technique for FV methods (Boris and Book, Zalesak):

- **compute tentative solution $\tilde{c}^{L}_{k,0}$ with low order flux $\hat{f}^{L}_{k+\frac{1}{2}}$ only,**

  compute $\tilde{c}^{L,max}_{k,0}$, $\tilde{c}^{L,max}_{k,0}$: maximum and minimum value of tentative solution averages over upwind elements
- **compute P^{+}_{k} and P^{-}_{k} as the sum of all antidiffusive fluxes into and away element k**
- **compute** $Q^{+}_{k} = \left(\tilde{c}^{L,max}_{k,0} - \tilde{c}^{L}_{k,0}\right)\Delta x_{k}, \quad Q^{-}_{k} = \left(\tilde{c}^{L}_{k,0} - \tilde{c}^{L,min}_{k,0}\right)\Delta x_{k}$
- **set** $R^{\pm}_{k} = \min\left(1, Q^{\pm}_{k}/P^{\pm}_{k}\right)$ **if** $P^{\pm}_{k} > 0$ $\;R^{\pm}_{k} = 0$ **if** $P^{\pm}_{k} = 0$

$$C^{\lim}_{k+\frac{1}{2}} = \min\left(R^{+}_{k}, R^{-}_{k+1}\right) \quad \textbf{if } A_{k+\frac{1}{2}} \geq 0$$
$$C^{\lim}_{k+\frac{1}{2}} = \min\left(R^{-}_{k}, R^{+}_{k+1}\right) \quad \textbf{if } A_{k+\frac{1}{2}} < 0$$

Examples of time discretizations

Strong stability preserving time discretizations (1)

- **General** $m-$**stage Runge Kutta method: partition of the time interval** $[0, T]$ **into** N_T **subintervals** $[t^n, t^{n+1}]$, **set** $\mathbf{d}_k^{(0)} = \mathbf{c}_k^n$, **then define**

$$\mathbf{d}_k^{(i)} = \sum_{l=0}^{i-1} \alpha_{il} \mathbf{d}_k^{(l)} + \beta_{il} \Delta t \mathbf{F}(\mathbf{d}^{(i)})_k \quad i = 1, \dots, m \qquad \mathbf{c}_k^{n+1} = \mathbf{d}_k^{(m)}$$

- **Strong stability preserving (SSP) method:**

$$\|\mathbf{d}_k^{(i)}\| \leq \|\mathbf{c}_k^n\| \quad \forall i = 1, \dots, m$$

- **Sufficient conditions for SSP:** $\alpha_{il} \geq 0$, $\beta_{il} \geq 0$ **+ stability restriction on** Δt **that depends on the Courant number**

Strong stability preserving time discretizations (2)

Optimal second order SSP Runge-Kutta scheme:

$$\mathbf{d}_k^{(1)} = \mathbf{c}_k^n + \Delta t \mathbf{F}(\mathbf{c}^n)_k,$$

$$\mathbf{c}_k^{n+1} = \frac{1}{2}\mathbf{c}_k^n + \frac{1}{2}\mathbf{d}_k^{(1)} + \frac{1}{2}\Delta t \mathbf{F}(\mathbf{d}^{(1)})_k$$

Optimal third order SSP Runge-Kutta scheme:

$$\mathbf{d}_k^{(1)} = \mathbf{c}_k^n + \Delta t \mathbf{F}(\mathbf{c}^n)_k,$$

$$\mathbf{d}_k^{(2)} = \frac{3}{4}\mathbf{c}_k^n + \frac{1}{4}\mathbf{d}_k^{(1)} + \frac{1}{4}\Delta t \mathbf{F}(\mathbf{d}^{(1)})_k,$$

$$\mathbf{c}_k^{n+1} = \frac{1}{3}\mathbf{c}_k^n + \frac{2}{3}\mathbf{d}_k^{(2)} + \frac{2}{3}\Delta t \mathbf{F}(\mathbf{d}^{(2)})_k$$

Implicit discretizations: TR-BDF2

- **Two stages method, (Bank et al, 1985), (Hosea and Shampine, 1996)**
- **One Trapezoidal Rule (Crank Nicolson) stage, one BDF2 stage:**

$$
\begin{aligned}
\mathbf{c}_k^{n+2\gamma} - \gamma \Delta t \mathbf{F}(\mathbf{c}^{n+2\gamma})_k &= \mathbf{c}_k^n + \gamma \Delta t \mathbf{F}(\mathbf{c}^n)_k \\
\mathbf{c}^{n+1} - \gamma_2 \Delta t \mathbf{F}(\mathbf{c}^{c+1})_k &= (1-\gamma_3)\mathbf{c}_k^n + \gamma_3 \mathbf{c}_k^{n+2\gamma}
\end{aligned}
$$

- **A-stable, L-stable for** $\gamma = 1 - \sqrt{2}/2 \quad \gamma_2 = \frac{1-2\gamma}{2(1-\gamma)}, \quad \gamma_3 = \frac{1-\gamma_2}{2\gamma}$
- **Full second order (in contrast to off centered Trapezoidal Rule)**

Degree adaptivity

The tools for degree adaptivity

- use Legendre basis: orthogonal and hierarchical basis, diagonal mass matrix
- representation for a model variable α over element k:

$$\alpha(x)\big|_k = \sum_{i=1}^{p(k)} \alpha_{k,i}\psi_{k,i}(x)$$

- root mean square norm $\|\alpha\|_2^2$ is given by $\mathcal{E}_k^{tot} = \sum_{i=1}^{p(k)} \alpha_{k,i}^2$
- $w_k^r = \sqrt{\frac{\alpha_{k,r}^2}{\mathcal{E}_k^{tot}}}$ relative weight of the $r-$ degree modes

A degree adaptivity algorithm

Given an error tolerance $\epsilon_k > 0$, for each element k, at each time step compute $w_{p(k)}$ and repeat following steps:

1) **if $w_{p(k)} \geq \epsilon_k$, then**
   1.1) **set $p(k) := p(k) + 1$**
   1.2) **set $\alpha_{k,p(k)} = 0$, exit the loop and go the next element**

2) **if instead $w_{p(k)} < \epsilon_k$, then**
   2.1) **compute $w_{p(k)-1}$**
   2.2) **if $w_{p(k)-1} \geq \epsilon_k$, exit the loop and go the next element**
   2.3) **else if $w_{p(k)-1} < \epsilon_k$, set $p(k) := p(k) - 1$ and go back to 2.1.**

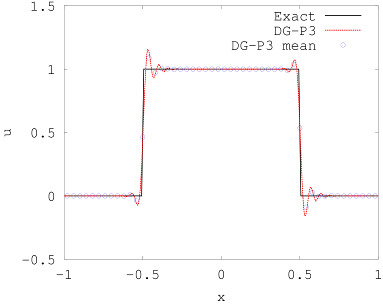
Numerical results

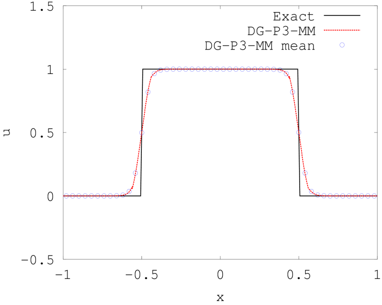
Comparison of limiters (1)

Linear advection problem with discontinuous initial datum. Errors on the maximum and minimum values of solution c and on its maximum and minimum mean values c_0

N_K	r	Limiter	$\max(c)-1$	$\min(c)+1$	$\max(c_0)-1$	$\min(c_0)+1$
100	1	None	6.9e-2	-6.9e-2	5.8e-2	-5.8e-2
100	1	FCT	1.0e-2	-1.0e-2	0	0
100	1	Minmod	5.2e-4	-5.7e-4	5.2e-4	5.7e-4
50	3	None	1.5e-1	-1.5e-1	8.9e-2	-8.9e-2
50	3	FCT	1.5e-2	-1.5e-2	-1.9e-4	1.9e-4
50	3	Minmod	0	0	0	0

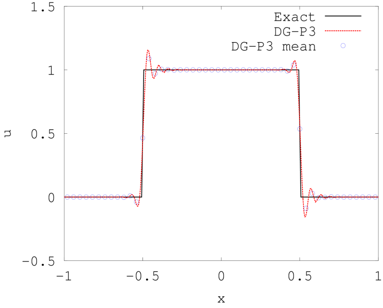
Comparison of limiters (2)

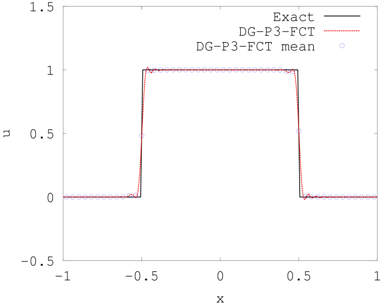




- **Linear advection test case with non smooth solution**
- **P3 solutions, no limiter (left) and minmod slope limiter (right)**
- **High order solution is fully monotone, discontinuity is smoothed**

Comparison of limiters (3)





- **Linear advection test case with non smooth solution**
- **P3 solutions, no limiter (left) and FCT flux limiter (right)**
- **High order solution is not fully monotone, but $P0$ component is fully monotone and discontinuity is less smoothed**

Model equations: shallow water

$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0$$

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -g\nabla\eta + f\mathbf{k} \times \mathbf{u}$$

- **Hyperbolic system in advective form, 2D framework for testing with all key features of geophysical scale dynamics.**
- **Semi-implicit, semi-Lagrangian DG discretization**
- $p-$**adaptive approach on quadrilateral meshes**

Efficiency gains

- **relative differences between adaptive (tolerance $\epsilon = 10^{-2}$) and nonadaptive solution:**

adaptivity	$l_1(h)$	$l_2(h)$	$l_\infty(h)$
static adaptivity	1.415×10^{-4}	3.314×10^{-4}	2.117×10^{-3}
static + dynamic	1.660×10^{-4}	3.419×10^{-4}	2.038×10^{-3}

- **dynamically $p-$adaptive solution CPU time: 33% of that for nonadaptive solution**

$$\frac{\#\textbf{gmres-it}(p^h = \textbf{adapt})}{\#\textbf{gmres-it}(p^h = \textbf{unif})} \approx 13\%, \quad \Delta^n_{dof} = \frac{\sum_{l=1}^{N}(p^n_l + 1)^2}{N(p_{max} + 1)^2} \approx 45\%.$$

Suggested reading

Reference books

- Whitham, G.B., Linear and nonlinear waves. Vol. 42. John Wiley & Sons, 1974
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- LeVeque, R. J. Finite volume methods for hyperbolic problems. Vol. 31. Cambridge University Press, 2002
- D. Kröner. Numerical schemes for conservation laws. Wiley-Teubner, 1997.
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- Cockburn, Bernardo, George E. Karniadakis, and Chi-Wang Shu. The development of discontinuous Galerkin methods. Springer Berlin Heidelberg, 2000.

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Work done in my group

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