Introduction to discontinuous finite element methods for hyperbolic equations, part (1)

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Outline of the lecture

Weak formulation and FE discretizations

Polynomial spaces for DG discretizations

Bases of polynomial spaces

Fully discrete formulation: quadrature rules

Key concepts introduced in the fifth lecture

- **Weak formulation of hyperbolic PDEs as basis for Continuous Galerkin and Discontinuous Galerkin discretizations**
- **Finite element polynomial spaces for DG discretizations**
- **Lagrange and Legendre basis for s polynomial spaces of DG discretizations**
- **Fully discrete formulation in space: quadrature rules**

Weak formulation and FE discretizations

Weak solution of conservation laws

$$\frac{\partial c}{\partial t} + \frac{\partial}{\partial x} f(c) = 0$$

The function $c(x,s)$ is a weak solution of the nonlinear conservation law for $s \in [0,t]$ with initial data $c_0(x)$ and boundary data $c(0,s) = g_0(s),\ c(L,s) = g_L(s)$ if for any $\phi \in \mathcal{C}^\infty([0,L]\times[0,t])$ it holds

$$\int_0^L c(x,t)\phi(x,t)\ dx + \int_0^t f(g_L(s))\phi(L,s)\ ds - \int_0^t f(g_0(s))\phi(0,s)\ ds$$
$$-\int_0^t \int_0^L \left[c\frac{\partial \phi}{\partial s} + f(c)\frac{\partial \phi}{\partial x}\right]\ dx\ ds = 0$$

- $\phi \in \mathcal{C}^\infty([0,L]\times[0,T])$: **functions of space and time, differentiable infinitely many times**
- **For discretization purposes, a concept of weak solution with respect to space variables only is introduced**

Weak solution of conservation laws, space dependent test functions

The function $c(x,t)$ is a weak solution (with respect to space) of the nonlinear conservation law for $t \in [0,T]$ with initial data $c_0(x)$ and boundary data $c(0,t)=g_0(s),\ c(L,t)=g_L(t)$ if for any $\phi \in \mathcal{C}^\infty([0,L])$ it holds

$$
\int_0^L \frac{\partial}{\partial t} c(x,t)\phi(x)\ dx + f(g_L(t))\phi(L) - f(g_0(t))\phi(0)
$$
$$
-\int_0^L f(c(x,t))\frac{\partial \phi}{\partial x}(x)\ dx = 0
$$

- $\phi \in \mathcal{C}^\infty([0,L])$: **test functions of space only, differentiable infinitely many times**
- **Interpretation: we do not care about discontinuities in time and only focus on discontinuities in space**
- **Also known as weak form of the original PDE**

Galerkin methods (1)

- **Assume** that weak form is only satisfied for all test functions in a **finite dimensional** function space
- **Assume** that the approximate solution belongs to a **finite dimensional** function space
- Let $\phi_i, \; i = 1, \dots, N$ be a **basis** for the finite dimensional space of test functions, let $\psi_j, \; j = 1, \dots, M$ be a **basis** for the finite dimensional space of solutions (ψ trial functions)
- The **approximate** solution will be of the form

$$c(x,t) \approx c_h(x,t) = \sum_{j=1}^{M} c_j(t)\psi_j(x)$$

Galerkin methods (2)

The approximate solution will be determined imposing the weak form of the equation for all $\phi_i, \ i = 1, \dots, N$:

$$\sum_{j=1}^{M} c_i'(t) \int_0^L \psi_j(x)\phi_i(x) \ dx + f(g_L(t))\phi_i(L) - f(g_0(t))\phi_i(0)$$

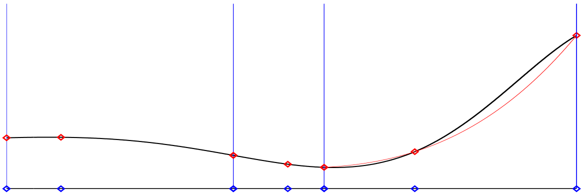
$$- \int_0^L f(c_h(x,t)) \frac{\partial \phi_j}{\partial x}(x) \ dx = 0$$

- **Finite dimensional system of ODEs whose unknowns are the solution coefficients $c_j(t)$**
- **Many ways to choose test and basis functions: many different Galerkin methods**

Galerkin methods (3)

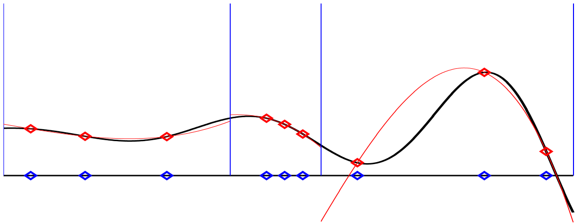
- For all methods: define mesh, decomposition of computational domain in subdomains called elements
- For all methods: test and basis functions depend on the mesh elements
- Test functions different from basis functions: Petrov - Galerkin methods
- Test functions identical to basis functions: standard Galerkin (or Ritz - Galerkin) methods (only case we will see in detail)

Continuous Galerkin method



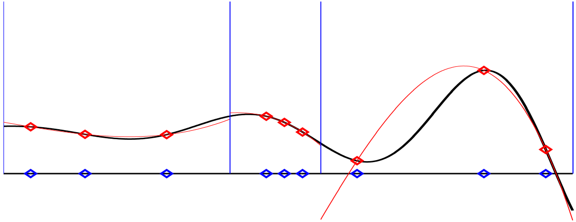
- Test and basis functions: functions that are a) polynomials when restricted to each element and b) continuous functions over the whole domain
- Excellent for elliptic and parabolic problems, less so for hyperbolic problems: no natural way to introduce upwinding
- Difficulties in using non conforming meshes and in introducing variable degree basis in more than one dimension

Discontinuous Galerkin method (1)



- **Test and basis functions: functions that are a) polynomials when restricted to each element, b) not required to be continuous over the whole domain**
- **Extension of finite volume methods: numerical fluxes are needed to define values at element interfaces**

Discontinuous Galerkin method (2)



- **Excellent** for hyperbolic problems, some problems with elliptic parabolic problems: artificial **stabilization** terms are needed, but complete theory now available
- Easy extension to **non conforming** meshes and to **variable degree** basis in **more than one** dimension
- High order approximations with **more compact stencil**: advantages for parallel implementations

Polynomial spaces for DG discretizations

Polynomial spaces (1)

- **Divide solution interval** $[0, L]$ **in** N_e **non overlapping elements** $[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$ $\;k = 1, \dots, N_e$ **of size** $\Delta x_k = x_{k+\frac{1}{2}} - x_{k-\frac{1}{2}}$, $h = \max\{\Delta x_k\}$
- **For each element** $k = 1, \dots, N_e$, **define nonnegative integer** $r = r(k)$ **and local polynomial space**

$$
\begin{aligned}
V_k^r &= \mathbb{P}^r([x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]) \\
&= \left\{ \text{polynomials on } [x_{k-\frac{1}{2}}, x_{i+\frac{1}{2}}] \text{ of degree} \leq \; r(k) \right\}
\end{aligned}
$$

- **Define** $p = \max\{r(k), k = 1, \dots, N_e\}$ **and global polynomial space**

$$
V_h^p = \left\{ \varphi \;\; : \varphi \in L^\infty([0, L]), \quad \varphi|_{[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]} \in V_k^r \right\}
$$

Polynomial spaces (2)

- V_h^p is a finite dimensional linear space, of dimension at most $N_e \times (p+1)$, V_k^r is a finite dimensional linear space, of dimension $r(k)+1$
- For the weak form of the equation to hold, it is sufficient that it holds for all $\phi_i, \ i = 1, \dots, N$ of a linear basis of V_h^p
- Since the functions of V_h^p are defined elementwise and each element of V_k^r does not depend on the functions of $V_l^r, \ k \neq l$ the weak form of the equation will be imposed element by element
- Increasing the value of p, a better approximation is achieved on smooth functions; if $p = 0$, first order finite volume methods are recovered

At the element edges

- **For** $\varphi \in V_h^p$ **define** $\varphi^{\pm}_{k\pm\frac{1}{2}} = \lim_{x\to x_{k\pm\frac{1}{2}}} \varphi(x), \;$ **with** $x \in [x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$
- **Define the average and the jump of the scalar function** $\varphi \in V_h^p$ **across the edge** $x_{k+\frac{1}{2}}$ **as**

$$\{\varphi\}_{k+\frac{1}{2}} = \frac{1}{2}\left(\varphi^{+}_{k+\frac{1}{2}} + \varphi^{-}_{k+\frac{1}{2}}\right) \qquad [[\varphi]]_{k+\frac{1}{2}} = \left(\varphi^{+}_{k+\frac{1}{2}} - \varphi^{-}_{k-\frac{1}{2}}\right)$$

- **Introduce a numerical flux function** $\hat{f}(a,b)$ **and define for** $c_h \in V_h^p$

$$f_{k+\frac{1}{2}}(t) = \hat{f}(c^{-}_{h,k+\frac{1}{2}}(t), c^{+}_{k+\frac{1}{2}}(t))$$

Discontinuos Galerkin methods, elementwise (1)

- **Assume that the weak form is satisfied for all test functions in** $V_k^r, \ k = 1, \dots, N_e$
- **Assume that the approximate solution restricted to** $[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$ **belongs to** V_k^r
- **Let** $\phi_{k,i}, \ i = 1, \dots, N(k)$ **be a basis for** V_k^r **: it follows that if** $c_h(x,t)|_{[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]} \in V_k^r$ **one has**

$$c(x,t)|_{[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]} \approx c_h(x,t)|_{[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]} = \sum_{j=1}^{N(k)} c_{k,j}(t)\phi_{k,j}(x)$$

Discontinuos Galerkin methods, elementwise (2)

The numerical solution will be determined by imposing that $c_h(x,t)|_{[x_{k-\frac{1}{2}},x_{k+\frac{1}{2}}]}$ satisfies the weak form of the equation for each $[x_{k-\frac{1}{2}},x_{k+\frac{1}{2}}]$. This is equivalent to:
for all $k=1,\dots,N_e$, for all basis functions $\phi_{k,i}(x), i=1,\dots,r(k)+1$

$$\sum_{j=1}^{N(k)} c'_{k,j}(t)\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \phi_{k,j}(x)\phi_{k,i}(x)\ dx$$
$$+\hat{f}(c^-_{h,k+\frac{1}{2}}(t),c^+_{h,k+\frac{1}{2}}(t))\phi_{k,i}(x_{k+\frac{1}{2}}) - \hat{f}(c^-_{h,k-\frac{1}{2}}(t),c^+_{h,k-\frac{1}{2}}(t))\phi_{k,i}(x_{k-\frac{1}{2}})$$
$$-\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f(c_h(x,t))\frac{\partial\phi_{k,i}}{\partial x}(x)\ dx = 0$$

Bases of polynomial spaces

Lagrange basis (1)

- **Let** $x_0^{(k)} < x_1^{(k)} < \cdots < x_p^{(k)}$ **nodes belonging to** $[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$
- **Lagrange basis functions on** $[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$ **are defined as**

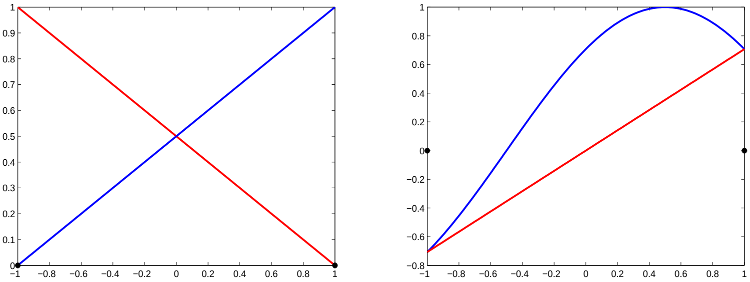
$$L_{k,i}(x) = \Pi_{j \neq i} \frac{x - x_j^{(k)}}{x_i^{(k)} - x_j^{(k)}} \qquad i = 0, \ldots, r(k),$$

- **Important property of Lagrange basis:**
  $L_{k,i}(x_i^{(k)}) = 1, \quad L_{k,i}(x_j^{(k)}) = 0 \quad \mathbf{i \neq j}$
- **As a result, if Lagrange basis is chosen,**

$$c_h(x_l^{(k)}, t) = \sum_{j=1}^{N(k)} c_{k,j}(t) L_{k,j}(x_l^{(k)}) = c_{k,l}(t)$$

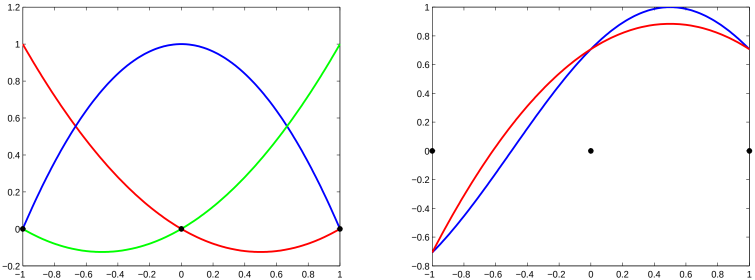
Coefficients $c_{k,l}(t)$ **are the nodal values of** $c_h(x, t)$ **at** $x_l^{(k)}$

Lagrange basis (2)



- **Lagrange basis with two nodes: linear approximation over each element**
- **Each of the two basis functions is a polynomial of degree 1**
- **Their linear combination yields a polynomial of degree 1 that approximates a given function**

Lagrange basis (3)



- **Lagrange basis with three nodes: quadratic approximation over each element**
- **Each of the three basis functions is a polynomial of degree 2**
- **Their linear combination yields a polynomial of degree 2 that approximates a given function**

Legendre basis (1)

- **Consider polynomial basis** $1, x - x_k, (x - x_k)^2, \ldots, (x - x_k)^p$ **on** $[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$
- **Orthogonalize by Gram-Schmidt procedure to obtain Legendre polynomials** $L_{k,i}(x), \quad i = 0, \ldots, p$
- **Can also be defined by the recurrence relation**

$$
\begin{aligned}
L_{k,i+1}(x) &= \frac{2i+1}{i+1}(x - x_k)L_{k,i}(x) - \frac{i}{i+1}L_{k,i}(x), \quad i = 1, 2, \ldots \\
L_{k,0}(x) &= 1, \\
L_{k,1}(x) &= x - x_k
\end{aligned}
$$

Legendre basis (2)

- **Legendre polynomials: orthogonal basis**

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} L_{k,p}(x) L_{k,q}(x)\ dx = 0 \qquad p \neq q$$

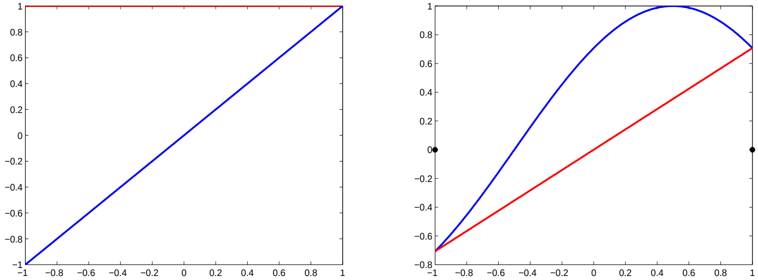
- **As a result, if Legendre basis is chosen,**

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} c_h(x,t) L_{k,i}(x)\ dx = \sum_{j=1}^{N(k)} c_{k,j}(t) \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} L_{k,j}(x) L_{k,i}(x)\ dx$$

- **Coefficients $c_{k,j}(t)$ are the modal coefficients of $c_h(x,t)$:**

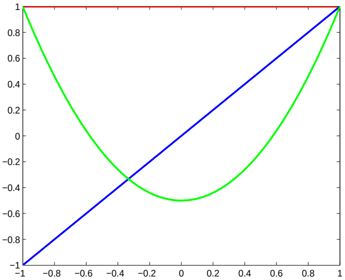
$$c_{k,j}(t) = \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} c_h(x,t) L_{k,j}(x)\ dx \Big/ \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} L_{k,j}(x)^2\ dx$$

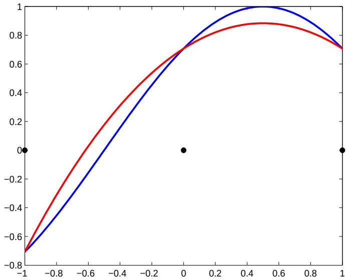
Legendre basis (3)



- **Legendre basis with two basis functions: linear approximation over each element**
- **First basis function is a polynomial of degree 0 (constant), second basis function is a polynomial of degree 1 (linear)**
- **Their linear combination yields a polynomial of degree 1 that approximates a given function**

Legendre basis (4)





- **Legendre basis with three basis functions: quadratic approximation over each element**
- **First basis function is a polynomial of degree 0 (constant), second basis function is a polynomial of degree 1 (linear), third basis function is a polynomial of degree 2 (quadratic)**
- **Their linear combination yields a polynomial of degree 2 that approximates a given function**

Fully discrete formulation: quadrature rules

Numerical integration formulae

- Galerkin methods require the computation of integrals like

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} \phi_{k,j}(x)\phi_{k,i}(x)\ dx \qquad \int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f(c_h(x,t))\frac{\partial \phi_{k,i}}{\partial x}(x)$$

- Accurate numerical integration methods are required
- Gaussian integration (quadrature) formulae are usually employed to achieve maximum accuracy

The master element

- **Each element $[x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}]$ is mapped onto the master element $[-1, 1]$ via the function**

$$x = x_{(k)}(\xi) = x_k + \frac{\Delta x_k}{2}\xi \qquad x_k = \frac{x_{k-\frac{1}{2}} + x_{k+\frac{1}{2}}}{2}$$

- **All integrals to be computed are reformulated by this change of variables as integrals on the master element**

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f(x)\,dx = \int_{-1}^{1} f(x_{(k)}(\xi))\, x'_{(k)}(\xi) d\xi = \int_{-1}^{1} f(x_k + \frac{\Delta x_k}{2}\xi)\frac{\Delta x_k}{2} d\xi$$

- **In this way, all numerical quadrature formulae only need to be introduced on $[-1, 1]$**

Gauss Legendre quadrature rules on $[-1, 1]$

- $$\int_{-1}^{1} f(\xi)d\xi \approx \sum_{i=1}^{q} f(\xi_i)w_i$$

  with special points $\xi_i, \ i = 1, \dots, q$ (Gaussian nodes) and special numbers $w_i, i = 1, \dots, q$ (Gaussian weights)
- **For Gauss - Legendre rules, nodes $\xi_i, \ i = 1, \dots, q$ are the zeros of the Legendre polynomial of degree $q: \quad L_q(\xi_i) = 0$**
- **For Gauss - Legendre rules, weights $w_i, \ i = 1, \dots, q$ are given by**

  $$w_i = \frac{2}{(1 - \xi_i^2)[L_q'(\xi_i)]^2}$$

- **Gauss Legendre rules with q nodes are exact for polynomials of degree up to $2q - 1$**

Two nodes Gauss Legendre quadrature rule

- **Gaussian nodes and weights:**

$$\xi_1 = -\frac{\sqrt{3}}{3} \qquad \xi_2 = \frac{\sqrt{3}}{3} \quad w_1 = 1 \qquad w_2 = 1$$

- **Approximation of integral on master element**

$$\int_{-1}^{1} f(\xi) d\xi \approx \left[f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)\right]$$

- **Approximation of integral on generic element**

$$\int_{x_{k-\frac{1}{2}}}^{x_{k+\frac{1}{2}}} f(x)\ dx \approx \left[f\left(x_k - \frac{\sqrt{3}}{3}\frac{\Delta x_k}{2}\right) + f\left(x_k + \frac{\sqrt{3}}{3}\frac{\Delta x_k}{2}\right)\right]\frac{\Delta x_k}{2}$$

- **This formula is sufficiently accurate for computation of solutions with P^1 (piecewise linear) approximation**

DG methods, fully discrete in space

For all $k = 1, \dots, N_e,$ **for all basis functions** $\phi_{k,i}(x), i = 1, \dots, r(k) + 1$

$$\sum_{j=1}^{N(k)} c'_{k,j}(t) m_{i,j} = -\hat{f}(c^-_{h,k+\frac{1}{2}}(t), c^+_{h,k+\frac{1}{2}}(t))\phi_{k,i}(x_{k+\frac{1}{2}})$$
$$+\hat{f}(c^-_{h,k-\frac{1}{2}}(t), c^+_{h,k-\frac{1}{2}}(t))\phi_{k,i}(x_{k-\frac{1}{2}}) + F_{k,i}(c_h(x,t))$$

$$m_{i,j} = \sum_{l=1}^{q} \phi_{k,j}(x_l^{(k)})\phi_{k,i}(x_l^{(k)}) w_l$$

$$F_{k,i}(c_h(x,t)) = \sum_{l=1}^{q} f(c_h(x_l^{(k)}, t)) \frac{\partial \phi_{k,i}}{\partial x}(x_l^{(k)}) w_l$$

Here $x_l^{(k)}, l = 1, \dots, q$ **denotes the Gaussian nodes mapped onto** $x_{k-\frac{1}{2}}, x_{k+\frac{1}{2}}$ **from** $[-1, 1]$

Key concepts introduced in the fifth lecture

- **Weak** form of hyperbolic PDEs leads to DG discretization
- Approximation by **polynomials** expressed in terms of Lagrange or Legendre basis
- Fully **discrete** formulation in space is obtained by numerical **quadrature** rules
- A **nonlinear** system of ODEs is obtained that **couples** the evolution of unknown coefficients in different elements